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Relativistic tunnelling time for electronic wave packets

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Abstract
We have analyzed the influence of the wave packet size in the relativistic tunnelling time τ and its uncertainty $\Delta\tau$ when it traverses a given potential barrier. The analytical expressions obtained for both magnitudes confirm that the size of the incident pulse has a significant effect on the tunnelling process. This effect is greater for short pulses, compared with the length of the barrier. For the evanescent zone, we have derived an analytical expression for τ with a good limit of validity. This expression constitutes a value tool to calculate the relativistic tunnelling time as a function of the incident wave packet with a good limit of validity. Superluminal propagation is found in this region but with a large value of the uncertainty $\Delta\tau$ compared with the tunnelling time itself. We can conclude that the probability of superluminal propagation is practically negligible in the evanescent region. In respect to the Klein zone, we have derived an analytical expression for τ that depends on the size of the incident wave packet and the width of the Lorentzian resonance Γ_r. This equation fits extremely well with our numerical results for Lorentzian resonances near the top of the Klein zone, where the overlap between them is negligible. As in the evanescent case, superluminal propagation is not likely to occur in the Klein region.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the last few decades the time it takes a particle or wave packet to traverse a given region has been widely discussed in the tunnelling time literature [1–6]. The time problem has been approached from many different points of view, mostly based on the non-relativistic Schrödinger's equation and, generally, leads to two characteristic times. All these approaches can consistently be formulated in terms of Green's function, based on measuring the spin rotation of an electron under a weak magnetic field acting on the region of interest. Both characteristic times, that correspond to the real and imaginary part of a complex tunnelling time, are not independent and are connected by Kramers–Kronig relations [7]. The imaginary component of the complex time, τ_2, is related to the transverse direction of propagation, while the real component of the complex time τ_1 is associated with the direction of propagation. The latter, in the limit of an opaque barrier or in the forbidden gap of a periodic system, yields to 'superluminal' results or faster than c tunnelling velocities. A great variety of theoretical and experimental work on this topic has been carried out during the last two decades [8–14].

Within conventional interpretations of quantum mechanics concepts, time appears only as a parameter and thus an expectation value of time is not defined [4]. As was pointed out by Pauli [15], the existence of such a time operator would imply an unbounded energy spectrum, given the uncertainty relation between time and energy. However, several authors have developed tunnelling time formalisms on the basis of quantum time operators. Miyamoto [16] introduced a positive operator valued measure (POVM) approach for the Aharanov–Bohm time operator, while Galapon [17] performed a theoretical development of a 'self-adjoint' time operator in a discrete spectrum quantum system. Other approaches avoid a time operator definition and deal with the tunnelling time problem in a different manner. Hara *et al* [18] used a real time stochastic process to derive tunnelling time expressions and Garcia–Calderon *et al* [19] carried out a passage time by means of the Feynman path approach.

Our group has studied the tunnelling time of electronic and photonic wave packets taking into account the specific form of the pulse [20]. We performed our calculations within a non-relativistic scheme based on the presence-time formalism, where the tunnelling time τ was obtained as an expectation value of the energy derivative operator $\widehat{T} = -\mathrm{i}\hbar\partial/\partial E$ in the energy representation. We found that the tunnelling time τ for an incident wave packet with Fourier components $\Phi_\mathrm{i}(E)$ that traverses a potential of height V_0 and length L can be written as

$$\tau = \frac{\int_0^{V_0} \mathrm{d}E\ |\widehat{t}(E)|^2\ |\Phi_\mathrm{i}(E)|^2\ \tau_1(E)}{\int_0^{V_0} \mathrm{d}E\ |\widehat{t}(E)|^2\ |\Phi_\mathrm{i}(E)|^2}, \tag{1.1}$$

where $|\widehat{t}(E)|^2$ is the transmission coefficient and $\tau_1(E)$ is the phase time, which corresponds to the energy derivative of the phase of the complex transmission amplitude $\varphi_\mathrm{t}(E)$ [1]

$$\tau_1(E) = \hbar\frac{\partial\varphi_\mathrm{t}(E)}{\partial E}. \tag{1.2}$$

Equation (1.1) allows us to study the dependence of the tunnelling time τ with the spatial width of an incident wave packet.

The Klein–Gordon equation has been widely used in the literature to derive relativistic tunnelling time expressions [21–24]. Other works deal with the Dirac equation to calculate different tunnelling times and derive relations between them within the relativistic framework. Winful *et al* [25] derived a general relation between the phase time and the dwell time for relativistic tunnelling particles and Cheng *et al* [26] studied the properties of group delay for Dirac particles travelling through a potential. Recently, Lunardi *et al* [27] dealt with the relativistic quantum mechanical problem of a Dirac particle tunnelling through two successive barriers and Bernardini [28] obtained the solutions for the relativistic tunnelling time of a one-dimensional potential, in the relativistic wave equation regime, for an incoming wave packet. Besides tunnelling time calculations, the Dirac equation has been modeled both theoretically and experimentally with two-level atoms in a harmonic trap [29–33].

For particles with rest mass m_0 and total energy E in the presence of a potential $V(z)$ restricted to a region $0 < z < L$, the Dirac equation is given by [25]

$$H_0\Psi = [-\mathrm{i}\hbar c\alpha_z\partial_z + \beta m_0c^2 + V(z)]\Psi = E\Psi, \tag{1.3}$$

where α_z and β are 4×4 matrices specified in terms of the Pauli matrices and the unit matrix I. On the other hand, Ψ is a 4-component wavefunction (Dirac bispinor) with two degrees of freedom for positive energy solutions (with spin up and down) and the other two for the corresponding negative energy solutions. The relativistic interaction of the Dirac particle incident on a barrier with height V_0 and length L can be divided into three cases. In the case that the potential barrier is low enough to satisfy $V_0 < E - m_0c^2$, the particle has enough energy to propagate over the potential barrier, as was discussed in [25]. No tunnelling phenomenon corresponds to this situation where non-evanescent wave propagation exists. When the potential barrier satisfies $E - m_0c^2 < V_0 < E + m_0c^2$, and also for positive Dirac particles, evanescent propagation occurs, by analogy with the non-relativistic tunnelling. The most dramatic case is when the potential barrier is strong enough to satisfy $V_0 > E+m_0c^2$. This is a transient phenomenon called Klein tunnelling, which has no non-relativistic equivalent. Here the particle is able to tunnel through the barrier without attenuation, a process mediated by spontaneous particle–antiparticle pair production [34–37]. Recently, quantum simulations of the Dirac equation with Bloch–Oscillating spinor atoms [38] and detailed analysis of the Klein tunnelling time in graphene [39–41] have been carried out.

Due to the increasing interest in the relativistic tunnelling in recent years, our group have developed the presence-time formalism to calculate analytically and numerically this magnitude and its uncertainty for wave packets that traverse a given potential barrier. With this method we take into account the specific form of the pulse and we can evaluate the dependence of both magnitudes with the size of the wave packet. In a non-relativistic formulation [20] we showed that the size of the pulse has a substantial effect on the tunnelling time and its uncertainty, an effect that is greater when the size of the pulse is of the same order of magnitude than the barrier length. Our aim in this paper is to study if this finite size effect is significant in the relativistic tunnelling time. As a consequence, relativistic formulas for both magnitudes, as a function of the wave packet size, are derived, formulas that should be taken into consideration when working with sort spatial pulses. The possible superluminal propagation can also be discussed by means of our relativistic formulation. To be consistent with this relativistic scheme and perform our tunnelling time calculations, we must redefine the time operator $\widehat{T}$ to be Lorentz-covariant. As shown by Olkhovsky [42], the following bilinear operator satisfies the latter condition

$$\widehat{T} = -\frac{i\hbar}{2}\frac{\overleftrightarrow{\partial}}{\partial E}, \tag{1.4}$$

where the expectation values are now evaluated as

$$\langle f|\widehat{T}|g\rangle = \left\langle f \left| \left(-\frac{i\hbar}{2}\frac{\partial}{\partial E}\right) g \right.\right\rangle + \left\langle \left(-\frac{i\hbar}{2}\frac{\partial}{\partial E}\right) f \right| g \right\rangle. \tag{1.5}$$

We will use equation (1.5) to derive analytical expressions for the relativistic tunnelling time τ and its uncertainty $\Delta\tau$.

The plan of the work is as follows. In section 2, we develop our formalism to the simplest case of a relativistic wave packet that propagates in free space. The relativistic tunnelling time and its uncertainty of a wave packet through a potential barrier, where evanescent propagation occurs, is discussed in section 3. Analytical formulas for both magnitudes as a function of the wave packet size are derived. In section 4, we deal with the Klein tunnelling time and obtain an expression for this time when the central energy of the incident pulse coincides with a single transmission resonance. This is an approximate formula with a good limit of validity near top of the Klein zone. In section 5, we present numerical results concerning the relativistic tunnelling time and its uncertainty, results that reveal the importance of the size of the wave packet in both magnitudes. Moreover, the possible superluminal propagation is discussed. Finally, we summarize our results in section 6.

2. Relativistic wave packet in free space

Let us first discuss the propagation of a relativistic wave packet Φ_{fr} in free space.At $t = 0$ this wave packet is peaked at z_0, has a spatial width Δz and moves to the right. The components of Φ_{fr} in the energy representation are given by the following Dirac bispinor [25]:

$$\Phi_{\mathrm{fr}}(z, E) = G(E)\begin{pmatrix}1\\0\\\eta\\0\end{pmatrix}\exp\left[\mathrm{i}k(z - z_0)\right], \tag{2.1}$$

where $G(E)$ is a normalized weight peaked at the total energy E_0 with an energy width ΔE and the subindex indicates free space. The relativistic wave number k and the parameter η depend on the total energy E as follows:

$$k(E) = \frac{1}{\hbar c}\sqrt{E^2 - m_0^2c^4}, \qquad \eta(E) = \sqrt{\frac{E - m_0c^2}{E + m_0c^2}}. \tag{2.2}$$

We can now derive an expression for the time it takes the free relativistic wave packet to travel from z_0 to z, evaluating the expectation value of the time operator $\widehat{T}$ (given by (1.4)) in the energy representation [20, 43]

$$\langle\widehat{T}(z)\rangle = \frac{1}{P}\int_0^\infty \mathrm{d}E\ \Phi_{\mathrm{fr}}^*(z, E)\left(-\frac{\mathrm{i}\hbar}{2}\frac{\overleftrightarrow{\partial}}{\partial E}\right)\Phi_{\mathrm{fr}}(z, E), \tag{2.3}$$

where P is the normalization factor

$$P(z) = \int_0^\infty \mathrm{d}E\ |\Phi_{\mathrm{fr}}(z, E)|^2. \tag{2.4}$$

Due to the fact that the operator $\widehat{T}$ is Hermitian [42] the imaginary part of its expectation value cancels, so equation (2.3) can be written, after some algebraic calculations, as

$$\left\langle\widehat{T}(z)\right\rangle = \frac{1}{P}\int_0^\infty \mathrm{d}E\ G^2(E)(1 + \eta^2(E))\tau_{\mathrm{rel}}(E), \tag{2.5}$$

where τ_{rel} is the time it takes a relativistic particle with total energy E to travel from z_0 to z

$$\tau_{\mathrm{rel}}(E) = \frac{E\ (z - z_0)}{c\sqrt{E^2 - m_0^2c^4}}. \tag{2.6}$$

In the limit of low velocities $v \ll c$ we can recover the non-relativistic result, taking into account the following relations for the term $(1 + \eta^2(E))$:

$$1 + \eta^2(E) = \frac{2E}{E + m_0c^2} \simeq \frac{2m_0c^2 + m_0v^2}{2m_0c^2 + \frac{1}{2}m_0v^2} \simeq 1, \tag{2.7}$$

and the relativistic crossing time τ_{rel}

$$\tau_{\mathrm{rel}} = \frac{E\ (z - z_0)}{c\sqrt{(E - m_0c^2)(E + m_0c^2)}} \simeq \frac{z - z_0}{v}; \tag{2.8}$$

hence, (2.5) reduces to

$$\left\langle\widehat{T}(z)\right\rangle \simeq \frac{1}{P}\int_0^\infty \mathrm{d}E\ G^2(E)\left[\frac{m_0\ (z - z_0)}{\sqrt{2m_0E}}\right], \tag{2.9}$$

where we have expressed the velocity v in terms of the non-relativistic energy of the particle E, that is, $v = (\sqrt{2m_0E})/m_0$. Equation (2.9) is in agreement with our previous result for the non-relativistic wave packet propagation in free space [20].

Non relativistic tunnelling time has been calculated in this section. We have only introduced the formalism that will be used throughout the paper and applied it to the simplest case of free propagation. In the next section, we evaluate the time it takes a relativistic wave packet to tunnel through a potential barrier as a function of its size.

3. Relativistic tunnelling time for a potential barrier

Our one-dimensional potential barrier of height V_0 is placed in the region $0 < z < L$. Upon imposing continuity of the spinor function across the interfaces $z = 0, L$, one has for the complex transmission amplitude $\widehat{t}(E)$ [25]:

$$\widehat{t}(E) = \frac{\exp[-\mathrm{i}k(E)L]}{\gamma(E)}, \tag{3.1}$$

where γ is given by

$$\gamma(E) = \cosh(\kappa L) - \frac{\mathrm{i}}{2}\left(\xi - \frac{1}{\xi}\right)\sinh(\kappa L), \tag{3.2}$$

and ξ and the decay constant for evanescent waves, κ, take the form, respectively

$$\xi(E) = \left(\frac{k}{\kappa}\right)\left(\frac{E - V_0 + m_0c^2}{E + m_0c^2}\right), \qquad \kappa(E) = \frac{1}{\hbar c}\sqrt{m_0^2c^4 - (V_0 - E)^2}. \tag{3.3}$$

The transmitted wave packet in the energy representation Φ_{tr} can be expressed in terms of the following Dirac bispinor (see equation (2.1)):

$$\Phi_{\mathrm{tr}}(z, E) = \widehat{t}(E)\, G(E) \begin{pmatrix} 1 \\ 0 \\ \eta \\ 0 \end{pmatrix} \exp(\mathrm{i}kz), \tag{3.4}$$

where we have chosen the phase in such a way that our origin of time is when the incident wave packet, propagating freely, would reach the left of the barrier.

So, the relativistic tunnelling time τ can be written as an expectation value of $\widehat{T}$ at $z = L$ [20]:

$$\tau = \langle \widehat{T}(L)\rangle = \frac{1}{P}\int_{E_1}^{E_{\mathrm{u}}} \mathrm{d}E\, \Phi_{\mathrm{tr}}^*(L, E)\left(-\frac{\mathrm{i}\hbar}{2}\frac{\overleftrightarrow{\partial}}{\partial E}\right)\Phi_{\mathrm{tr}}(L, E), \tag{3.5}$$

and the integration limits correspond to $E_1 = V_0 - m_0c^2$ and $E_{\mathrm{u}} = V_0 + m_0c^2$, as required for the evanescent tunnelling zone. P is again a normalization factor given by

$$P = \int_{E_1}^{E_{\mathrm{u}}} \mathrm{d}E\, |\Phi_{\mathrm{tr}}(L, E)|^2. \tag{3.6}$$

Introducing the relativistic spinor (3.4) into (3.5) and after performing some algebraic operations, we arrive at the following expression for the real part of $\langle\widehat{T}\rangle$:

$$\tau = \frac{1}{P}\int_{E_1}^{E_{\mathrm{u}}} \mathrm{d}E\, G^2(E)|\widehat{t}(E)|^2(1 + \eta^2(E))\tau_1(E), \tag{3.7}$$

where τ_1 is the phase time.

Our main result concerning the relativistic tunnelling time, equation (3.7), is quite difficult to evaluate analytically. In order to obtain an analytical approximation for τ with a good limit of validity, we consider $G(E)$ as a Gaussian wave packet centered at E_0 of small energy width ΔE and expand $|\widehat{t}(E)|^2$ and $\tau_1(E)$ in Taylor series up to fourth order near E_0. The reason for this high-order expansion will be discussed in the numerical results section. Neglecting high-order derivatives of the characteristic times τ_1 and τ_2, (3.7) reduce to the following approximate expression:

$$\tau \simeq \tau_1(E_0) + \left(\frac{1}{\hbar^2}\right)[\tau_2(E_0)\,\widetilde{\tau}_1(E_0)]\,(\Delta E)^2 + \left(\frac{1}{\hbar^4}\right)[\tau_2(E_0)\,\widetilde{\tau}_1(E_0)\,\widetilde{\tau}_2(E_0)]\,(\Delta E)^4 + \left(\frac{1}{\hbar^6}\right)\left[\tau_2^5(E_0)\,\widetilde{\tau}_1(E_0)\right](\Delta E)^6, \tag{3.8}$$

where, as mentioned in the introduction, τ_2 is the tunnelling time component related to the transverse direction of propagation [1]

$$\tau_2(E) = \hbar\,\frac{\partial \ln|\widehat{t}(E)|}{\partial E}, \tag{3.9}$$

and $\widetilde{\tau}_{1,2}$ are the derivatives of $\hbar\tau_{1,2}$ with respect to energy, respectively. We will check the validity of (3.8) in section 5.

Our formalism also allows us to calculate the uncertainty of the relativistic tunnelling time $\Delta\tau$ at $z = L$ via the time operator $\widehat{T}$ [20]:

$$\Delta\tau = [\langle\widehat{T}^2\rangle - \langle\widehat{T}\rangle^2]^{1/2}, \tag{3.10}$$

where the brackets indicate, as usual, expectation values of the corresponding operators over the total energy E. After some tedious calculations involving the bilinear operator $\widehat{T}$, we obtain for the uncertainty of the relativistic tunnelling time at L

$$\Delta\tau = \left[\frac{1}{P}\int_{E_l}^{E_u} \mathrm{d}E\big[G^2|\widehat{t}|^2(1+\eta^2)\big(\tau_1^2 - \widetilde{\tau}_2 - \widetilde{\tau}_G - (\tau_2+\tau_G)^2\big) - G^2|\widehat{t}|^2\eta^2(\tau_\eta^2 + \widetilde{\tau}_\eta + 2\tau_\eta(\tau_2+\tau_G))\big] - \tau^2\right]^{1/2}, \tag{3.11}$$

where the parameters τ_G and τ_η are defined as

$$\tau_G(E) = \hbar\frac{\partial \ln G(E)}{\partial E}, \qquad \tau_\eta(E) = \hbar\,\frac{\partial \ln \eta(E)}{\partial E}, \tag{3.12}$$

and $\widetilde{\tau}_{G,\eta}$ correspond to the derivatives of $\hbar\tau_{G,\eta}$ with respect to the energy, respectively.

The latter expression for the uncertainty $\Delta\tau$ is also quite difficult to express analytically, so numerical methods are required to evaluate it. In section 5 we compare the numerical results given by (3.11) with the following second-order approximation of $\Delta\tau$ obtained by our group [20]:

$$\Delta\tau \simeq \frac{\hbar}{\sqrt{2}\,\Delta E}, \tag{3.13}$$

that gives relatively good results for different sizes of the incident relativistic wave packet, as we will show in the following section.

4. Klein tunnelling time

For strong barriers such that $V_0 > E + m_0c^2$ we encounter the phenomenon of Klein tunnelling which has no equivalence in the non-relativistic Schödinger's equation. In this critical case, the complex transmission amplitude $\widehat{t}(E)$ is now given by [25]

$$\widehat{t}(E) = \frac{\exp[-\mathrm{i}k(E)L]}{\gamma'(E)}, \tag{4.1}$$

where γ' takes the form

$$\gamma'(E) = \cos(\kappa' L) + \frac{\mathrm{i}}{2}\left(\xi' + \frac{1}{\xi'}\right)\sin(\kappa' L), \tag{4.2}$$

and the parameters ξ' and κ' can be expressed, respectively, as

$$\xi'(E) = \left(\frac{k}{\kappa'}\right)\left(\frac{V_0 - E - m_0c^2}{E + m_0c^2}\right), \qquad \kappa'(E) = \frac{1}{\hbar c}\sqrt{(V_0 - E)^2 - m_0^2c^4}. \tag{4.3}$$

We can easily find the transmission coefficient for the potential barrier $|\widehat{t}(E)|^2$ in the Klein regime via equation (4.1):

$$|\widehat{t}(E)|^2 = \left[\cos^2(\kappa' L) + \frac{1}{4}\left(\xi' + \frac{1}{\xi'}\right)^2 \sin^2(\kappa' L)\right]^{-1}, \tag{4.4}$$

which consists of a set of Lorentzian resonances for the resonant energies E_{r} [35]:

$$E_{\mathrm{r}} = V_0 - \left[\left(\frac{n\pi\hbar c}{L}\right)^2 + m_0^2c^4\right]^{1/2}, \qquad n = 0, 1, 2, \ldots . \tag{4.5}$$

So, a relativistic particle with incident energy E_{r} can totally be transmitted through the potential barrier, in other words, the barrier becomes practically transparent [28]. One encounters this transmission coefficient $|\widehat{t}(E)|^2$ for one-dimensional periodic structures with alternating indexes of refraction, as recently studied by our group in a detailed slow-light analysis [44].

Let us now obtain an analytical expression for the Klein tunnelling time τ as a function of the size of the incident wave packet. To this aim, we consider a single resonance E_{r} and assume a Gaussian wave packet of energy width ΔE centered at this resonant energy. For one Lorentzian resonance of width Γ_{r}, the transmission coefficient $|\widehat{t}(E)|^2$ and its phase $\varphi_{\mathrm{t}}(E)$ can written as follows [45]:

$$|\widehat{t}(E)|^2 = \frac{1}{1 + [a_{\mathrm{r}}(E - E_{\mathrm{r}})]^2}, \qquad \varphi_{\mathrm{t}}(E) = -\arctan\left(\frac{a_{\mathrm{r}}^{-1}}{E - E_{\mathrm{r}}}\right), \tag{4.6}$$

where $a_{\mathrm{r}} = 2/\Gamma_{\mathrm{r}}$. We can easily prove that the phase time τ_1, equation (1.2), can be expressed in terms of $|\widehat{t}(E)|^2$

$$\tau_1(E) = \hbar a_{\mathrm{r}}\, |\widehat{t}(E)|^2, \tag{4.7}$$

so, for the resonant energy E_{r}, the phase time is proportional to the resonance lifetime, that is, $\tau_1 = (2\hbar)/\Gamma_{\mathrm{r}}$.

The average given by (3.7) is also valid for the Klein tunnelling time τ with new integration limits, $E_{\mathrm{l}} = 0$ and $E_{\mathrm{u}} = V_0 - m_0c^2$. So, introducing the previous expressions of $|\widehat{t}(E)|^2$ and $\tau_1(E)$ into (3.7) and evaluating the corresponding integral, we find the following result for τ:

$$\tau = \hbar\left(\frac{a_{\mathrm{r}}}{2} + \frac{\exp[-(a_{\mathrm{r}}\Delta E)^{-2}]}{\sqrt{\pi}\,\mathrm{erfc}[(a_{\mathrm{r}}\Delta E)^{-1}]\Delta E} - \frac{1}{a_{\mathrm{r}}(\Delta E)^2}\right). \tag{4.8}$$

Having a close look into (4.8) one can deduce that, as ΔE tends to 0, that is, large spatial wave packets, the Klein tunnelling time reduces to $(2\hbar)/\Gamma_{\mathrm{r}}$, in consistency with the phase time τ_1 for a relativistic particle with resonant energy E_{r} (see equation (4.7)). In the next section we will verify the validity of (4.8) for different sizes of the incident Gaussian wave packet.

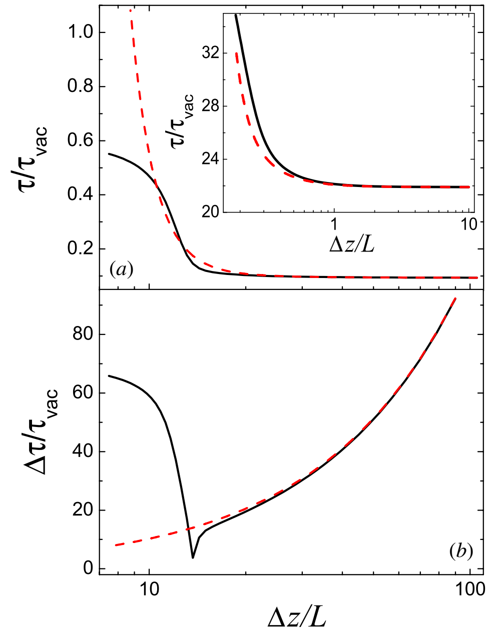

Figure 1. (*a*) Normalized tunnelling time $\tau/\tau_{\rm vac}$, versus the ratio $\Delta z/L$, for an incident Gaussian wave packet with central velocity $v_0 = 0.974c$. The barrier parameters are $V_0 = 65\,000$ and $L = 3$, in au. The solid line corresponds to the numerical results based on (3.7) while the dashed curve represents the fourth-order approximation, equation (3.8). The inset shows the non-relativistic case where now $v_0 = 0.020c$, $V_0 = 4$ and $L = 10$, in au. (*b*) Normalized uncertainty $\Delta\tau/\tau_{\rm vac}$ versus $\Delta z/L$ for the same Gaussian wave packet and potential barrier as shown in figure 1. The solid curve represents the numerical results obtained via (3.11) while the dashed curve corresponds to the second-order approximation, equation (3.13).

5. Numerical results

In this section we present some numerical results concerning the relativistic tunnelling time τ and its uncertainty $\Delta\tau$ as a function of the size of the incident wave packet Δz. The tunnelling time, associated with evanescent waves, and the Klein tunnelling time are both investigated. As we will see, the approximate results for τ and $\Delta\tau$ have a good limit of validity for a wide range of wave packet sizes. We use in all our work atomic units (au).

In figure 1(*a*), we represent the normalized tunnelling time, $\tau/\tau_{\rm vac}$, versus the ratio $\Delta z/L$, for an incident Gaussian wave packet with central energy $E_0 = 83\,768.9$ (corresponding to a central velocity of $v_0 = 0.974c$) that traverses a potential barrier of height $V_0 = 65\,000$ and length $L = 3$, in au. The parameter $\tau_{\rm vac}$ is the crossing time of the barrier at the vacuum speed of light. Under these assumptions, the relativistic tunnelling condition $E - m_0c^2 < V_0 < E + m_0c^2$ is satisfied ($c = 137.036$ au). The solid curve represents the results obtained via (3.7) and the dashed curve corresponds to the fourth-order approximation, equation (3.8). For all the represented values of Δz, the normalized tunnelling time is less than 1, so superluminal propagation is expected. We will discuss this situation later once obtained numerical results for the uncertainty of the tunnelling time $\Delta\tau$. We can observe that the dashed curve fits the numerical results relatively well up to the values of Δz of the order of nine times the barrier length. For lower values of Δz, the transmission coefficient cannot be replaced by a polynomial

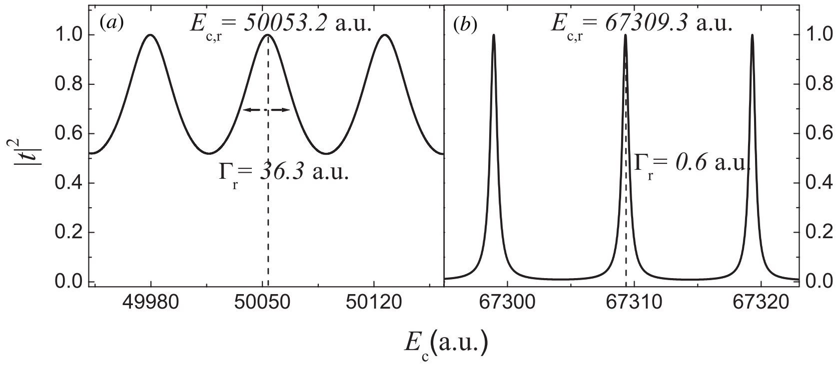

Figure 2. Transmission coefficient $|\hat{t}|^2$ for a strong potential barrier of height $V_0 = 105\,000$ and length $L = 5.0$, in au, for relativistic kinetic energies E_c (*a*) near 50 000 au and (*b*) near the top of the Klein region.

approximation and more complex terms are needed to improve the results. The inset shows the normalized tunnelling time in the non-relativistic case, where now $E_0 = 18\,782.8$ (central velocity of $v_0 = 0.020c$). The barrier parameters are $V_0 = 4$ and $L = 10$, in au. In this case, the fourth-order approximation is valid up to values of Δz of the order of L. These results agree relatively well to that obtained by our group [20].

In relation to the uncertainty of the relativistic tunnelling time $\Delta\tau$, we show in figure 1(*b*) the normalized uncertainty, $\Delta\tau/\tau_0$, versus $\Delta z/L$ for the same Gaussian wave packet and potential barrier as the previous case. The solid curve corresponds to the numerical results obtained via (3.11) and the dashed curve to the second-order approximation, equation (3.13). One notes that the second-order approximation has a good limit of validity up to the values of Δz similar to 14 times the barrier length. To describe analytically the sudden increase of $\Delta\tau$ for short wave packets we must consider more terms in our approximation. However, the main consequence of our previous results is that the uncertainty $\Delta\tau$ is much higher than the relativistic tunnelling time itself τ. So, we can conclude that the probability of superluminal propagation is practically negligible in the evanescent region.

In order to study numerically the Klein tunnelling time, we consider a strong potential barrier of height $V_0 = 105\,000$ and length $L = 5.0$, in au. The Klein region, where $V_0 > E + m_0c^2$, corresponds to relativistic kinetic energies satisfying $E_c < 67\,442.3$ au. We have represented in figure 2(*a*) the transmission coefficient $|\hat{t}|^2$ versus E_c for kinetic energies similar to 50 000 au, while in figure 2(*b*) we have shown the same parameter for E_c near the top of the Klein region. One can observe that the overlap between Lorentzians is negligible near the top, that is, complete resonances occur in this region.

Once analyzed the transmission coefficient of our barrier in the Klein zone, we plot in figure 3(*a*) the normalized Klein tunnelling time τ/τ_{vac} versus the ratio $\Delta z/L$ for an incident Gaussian wave packet centered at the resonance $E_{c,r} = 50053.2$ au (with a central velocity of $v_0 = 0.96c$). The solid line represents our numerical calculations obtained via (3.7), while the dashed line are the approximate results given by (4.8). For long spatial wave packets, equivalently, short values of the energy width ΔE, the normalized Klein tunnelling time saturates to the value 1.61. This corresponds to the resonance lifetime $(2\hbar)/\Gamma_r$, where $\Gamma_r = 36.3$ au in this case. In figure 3(*b*) we show the same parameters for a Gaussian wave packet centered at $E_{c,r} = 67\,309.3$ au. For this resonance, the approximate calculations fit relatively well the numerical results for practically all sizes of the incident wave packet. The

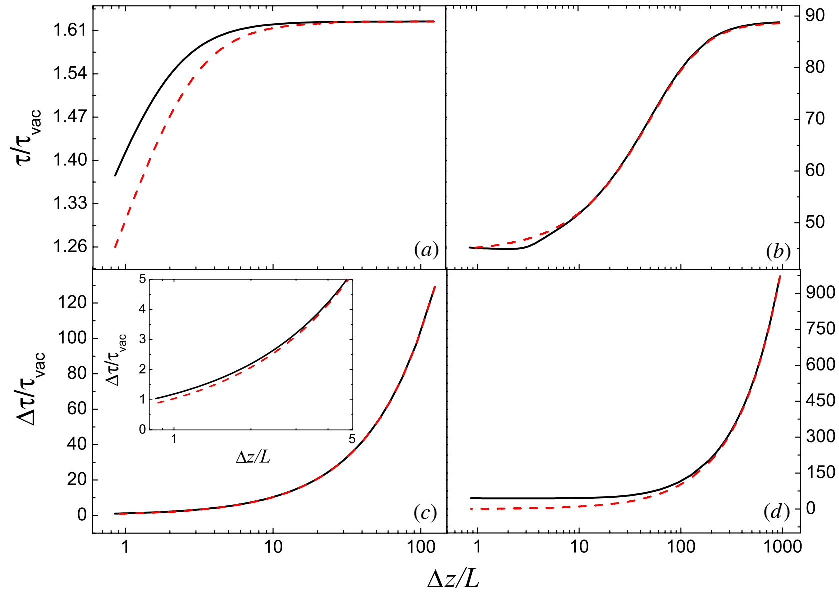

Figure 3. Normalized Klein tunnelling time τ/τ_{vac} versus the ratio $\Delta z/L$, for an incident Gaussian wave packet centered at (*a*) $E_{\text{c,r}} = 50053.2$ and (*b*) $E_{\text{c,r}} = 67\,309.3$, in au. The solid line corresponds to the numerical calculations obtained via (3.7), while the dashed lines are the approximate results given by (4.8). The normalized uncertainties $\Delta\tau/\tau_{\text{vac}}$ are shown in figures (*c*) (where $E_{\text{c,r}} = 50053.2$ au) and (*d*) ($E_{\text{c,r}} = 67\,309.3$ au). The solid curves represent the numerical results via (3.11) and the dashed curve to the second-order approximation, equation (3.13). The inset in figure (*c*) corresponds to the same curve as the main part, and has been included for the sake of clarity.

reason is that there exist complete Lorentzian resonances near the top of the Klein region where our approximation, given by (4.8), is more suitable than in the overlapped Lorentzians region.

One observes that, in both regions, τ/τ_{vac} is always greater than 1 so, superluminal propagation is not likely to occur. To confirm this matter, we plot the normalized uncertainty $\Delta\tau/\tau_{\text{vac}}$ for the overlapped Lorentzians region (figure 3(*c*)) and near the top of the Klein region (figure 3(*d*)). The solid curves represent the numerical results via (3.11) and the dashed curve to the second-order approximation, equation (3.13). The inset in figure 3(*c*) corresponds to the same curve as the main part, and has been included for the sake of clarity. As in the evanescent case, $\Delta\tau/\tau_{\text{vac}}$ is much higher or of the same order of magnitude than the Klein tunnelling time itself, so we conclude that the probability of superluminal propagation is practically negligible in the Klein region.

6. Conclusions

We have analyzed the influence of the wave packet size in the relativistic tunnelling time and its uncertainty when it traverses a given potential barrier. The analytical expressions derived for both magnitudes confirm that the size of the incident pulse has a significant effect on the tunnelling process. This effect is greater for short pulses, compared with the length of the barrier. For the evanescent zone, the approximate expression given by equation (3.8) constitutes a value tool to calculate the relativistic tunnelling time as a function of the incident wave packet with a good limit of validity. Superluminal propagation is found in this region but with a large value of the uncertainty $\Delta\tau$ compared with the tunnelling time itself. We can conclude that the probability of superluminal propagation is practically negligible in the evanescent region. In respect to the Klein zone, we have derived a useful expression for tunnelling time τ (see equation (4.8)) that depends on the size of the incident wave packet and the width of the Lorentzian resonance Γ_r. This equation fits extremely well our numerical results for Lorentzian resonances near the top of the Klein zone, where the overlap between them is negligible. As in the evanescent case, superluminal propagation is not likely to occur in the Klein region.

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