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Transmission of waves through one-dimensional random layered systems

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Abstract. A convenient formalism is developed that allows one to express the transmission coefficient of a wave propagating in a one-dimensional disordered structure through the determinant $T_N = |D_N|^{-2}$, which depends on the amplitudes of reflection of a single scatterer only. It is shown that the density of states averaged over the sample as well as the spectrum of surface and volume waves in such a layered system may also be represented by the determinant D_N.

1. Introduction

Transmission of a particle through a random one-dimensional medium is a problem studied repeatedly since Mott's pioneer work on electron behaviour in a random potential [1]. However, the methods available at present permit us to study analytically only the case of a 'white noise' type potential [2–5] in the weak scattering limit. Another method, that of the 'transfer matrix', allows us to carry out numerical calculations effectively [6–8]. A new method [9, 10] of investigating the one-dimensional random chain is the 'determinant' method, which allows us to solve the problem of electron transmission through a random system exactly, without actually determining the eigenfunctions of the electron. It was shown that, with the help of this method, a sufficiently complete description of electron behaviour in a random potential, which consists of a sum of δ-potentials, distributed randomly or with arbitrary strength, may be obtained. This 'determinant' is built up of the transmission and reflection coefficients of the scatterer, and not by the potential strength. The aim of this paper is to carry out a generalization of the method for the transmission of a free particle through a layered medium, which is characterized by random parameters of the layers. The particle may also be subject to an external electric field (see sections 2 and 3).

In section 4 we consider the transmission of light through a random layered structure. Our method makes it possible to explain the dependence of the localization length on the angle of incidence for different polarizations of light. It is necessary to note that the transmission of light through the system, which consists of periodically arranged thin metal films (the dielectric constant is complex), is the experimental realization of the Lloyd model. In section 5 we illustrate the connection between 'the determinant' and the energy spectrum of elementary excitations and obtain the polariton and plasmon

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spectra in layered structures. In section 6, this method is further applied to calculate the resistance of layered structures, consisting of alternating layers of a normal metal (or semiconductor) and a ferromagnet. Here the degree of polarization of the electron wave passing through such a structure is calculated. We consider cases where the magnetization vectors in the layers are parallel to each other, but arbitrary in quantity, or remain collinear, having different signs. In section 7 it is shown how, locally, also the average densities of states may be calculated with the help of the 'determinant' method.

2. Derivation of the main relations

Let us consider $(N-1)$ layers labelled $n = \{1, \ldots, N-1\}$ between two semi-infinite media. The positions of the boundaries of the nth layer are given by x_n and x_{n+1} respectively, and we associate a discontinuity in the potential $V_n(x)$ with each boundary.

We assume that a plane wave is incident from the left onto the boundary at $x = x_1$ and evaluate the amplitude of the reflected wave and the wave propagating in the semi-infinite media for $x \geq x_N$.

A convenient mathematical method, allowing us to take into account multiple interfaces consistently and exactly without the use of the perturbation theory, is the surface Green function (GF) method, proposed in [11, 12] for the investigation of the electron energy spectrum in inhomogeneous systems. This method has been applied to various problems in solid state physics before [13–18]. In this method the GF is evaluated first when one boundary between two media is available. The case of two boundaries is solved using the GF for one boundary. Therefore we solve the problem iteratively with $n+1$ boundaries, considering the solution with n boundaries to be known.

Let us first discuss the contact of two semi-infinite media: on the left of the boundary at x_1 $(x < x_1)$ the potential energy of the electron is $V_0(x)$, and on the right $V_1(x)$ $(x > x_1)$. Let us suppose that the one-dimensional electron GF $G_n^{(0)}(x, x'; E)$ $(n = 0, 1)$ for each medium are known, when the media are infinite. The following equation holds:

$$[-\partial^2/\partial x^2 + V_n(x) - E]G_n^{(0)}(x, x'; E) = \delta(x - x') \tag{1}$$

where $\hbar = 2m_0 = 1$ and m_0 is the free-electron mass.

We shall consider below constant potentials V_n with arbitrary strength for the two cases when we have free electrons or an additional homogeneous electrical field present $(V_{\text{tot}} = V_n + eEx)$. In the following the energy parameter E will be omitted in the argument of the GF. The upper index (l) of GF (in equation (1) the index $l = 0$) indicates that the GF is calculated in the presence of l boundaries. The lower index of the GF labels the interval for which the GF is valid. $G_0^{(1)}$ and $G_1^{(1)}$ are expressed in the form

$$G_{(0)}^{(1)}(x, x') = G_0^{(0)}(x, x') - r_{01}\frac{G_0^{(0)}(x, x_1)G_0^{(0)}(x_1, x')}{G_0^{(0)}(x_1, x_1)} \qquad x, x' \leq x_1 \tag{1a}$$

$$G_1^{(1)}(x, x') = G_1^{(0)}(x, x') - r_{10}\frac{G_1^{(0)}(x, x_1)G_1^{(0)}(x_1, x')}{G_1^{(0)}(x_1, x_1)} \qquad x, x' \geq x_1. \tag{1b}$$

To define the quantities r_{10} and r_{01}, let us write the condition of continuity for $G_0^{(1)}(x_1, x_1)$ and $G_1^{(1)}(x_1, x_1)$ and the equality of currents at the boundary $x = x_1$:

$$G_0^{(1)}(x_1, x_1) = G_1^{(1)}(x_1, x_1) \tag{2a}$$

$$\partial[G_0^{(1)}(x_1, x_1) - G_1^{(1)}(x_1, x_1)]/\partial x_1 = 0. \tag{2b}$$

The condition (2*b*) may also be written in the form:

$$\dot{G}_0^{(1)}(x_1 - 0, x_1) - \dot{G}_1^{(1)}(x_1 + 0, x_1) = 1. \tag{2c}$$

Here the dot signifies the derivative with respect to the first argument, and it is necessary to distinguish right-side and left-side derivatives of GF due to discontinuity:

$$\dot{G}_n^{(0)}(x \mp 0, x) = \pm\tfrac{1}{2} + \tfrac{1}{2}\,\partial G_n^{(0)}(x, x)/\partial x. \tag{3}$$

Solving equations (2), we obtain the following expressions for the coefficients r_{01} and r_{10} ($G_n^{(0)} \equiv G_n^{(0)}(x_1, x_1)$, $n = 0, 1$):

$$\begin{aligned} r_{01} &= \frac{G_0^{(0)}\dot{G}_1^{(0)}(x_1 + 0, x_1) - G_1^{(0)}\dot{G}_0^{(0)}(x_1 + 0, x_1)}{G_0^{(0)}\dot{G}_1^{(0)}(x_1 + 0, x_1) - G_1^{(0)}\dot{G}_0^{(0)}(x_1 - 0, x_1)} \\ r_{10} &= \frac{G_0^{(0)}\dot{G}_1^{(0)}(x_1 - 0, x_1) - G_1^{(0)}\dot{G}_0^{(0)}(x_1 - 0, x_1)}{G_0^{(0)}\dot{G}_1^{(0)}(x_1 + 0, x_1) - G_1^{(0)}\dot{G}_0^{(0)}(x_1 - 0, x_1)}. \end{aligned} \tag{4}$$

The quantity r_{01} (r_{10}) is the amplitude of the reflection of the electron propagating from the region 0 into 1 (1 into 0), and the squares of the moduli r_{01} and r_{10} are reflection coefficients [12]. If the medium is homogeneous along the axis x, i.e.

$$\partial G_{0,1}^{(0)}(x, x)/\partial x = 0$$

equation (4) for r_{01} and r_{10} may be presented in the form:

$$r_{01} = -r_{10} = (G_0^{(0)} - G_1^{(0)})/(G_0^{(0)} + G_1^{(0)}). \tag{4a}$$

Let us add another boundary from the right, at the point x_2, i.e. we consider a film placed between two semi-infinite media. As in deriving equation (4), let us suppose that GF in different parts of such a system in the absence of boundaries, i.e. $G_n^{(0)}(x, x')$ ($n = 0, 1, 2$), are known. To evaluate the amplitude of reflection and the explicit form of GF on the left and right of the new boundary, let us note that for $x, x' \leqslant x_2$, equations (1*a*) and (1*b*) will perform the role of an initial GF. As a result the GF in the regions $x, x' \geqslant x_2$ and $x_1 \leqslant x, x' \leqslant x_2$ will have the form:

$$G_2^{(2)}(x, x') = G_2^{(0)}(x, x') - R_{21}^{(2)}\frac{G_2^{(0)}(x, x_2)G_2^{(0)}(x_2, x')}{G_2^{(0)}(x_2, x_2)} \qquad x, x' \geqslant x_2 \tag{5a}$$

$$G_1^{(2)}(x, x') = G_1^{(1)}(x, x') - R_{12}^{(2)}\frac{G_1^{(1)}(x, x_2)G_1^{(1)}(x_2, x')}{G_1^{(1)}(x_2, x_2)} \qquad x_1 \leqslant x, x' \leqslant x_2 \tag{5b}$$

where $G_1^{(1)}(x, x')$ is expressed by equation (1*b*). The generalized quantity $R_{nm}^{(l)}$ is the reflection coefficient from the boundary between nth and mth layer with l boundaries in the sample, when the wave propagates from the nth layer into the mth layer.

Having written the condition of continuity for the new GF equations (5*a*) and (5*b*) at the point $x = x_2$, analogous to equation (2), we evaluate coefficients $R_{21}^{(2)}$ and $R_{12}^{(2)}$ [15].

It is apparent that the structure $R_{21}^{(2)}$ ($R_{12}^{(2)}$) will be the same as that of r_{10} (r_{01}) (see equation (4)):

$$R_{21}^{(2)} = \frac{G_1^{(1)}\dot{G}_2^{(0)}(x_2-0,x_2) - G_2^{(0)}\dot{G}_1^{(1)}(x_2-0,x_2)}{G_1^{(1)}\dot{G}_2^{(0)}(x_2+0,x_2) - G_2^{(0)}\dot{G}_1^{(1)}(x_2-0,x_2)}$$

$$= \frac{r_{21} + \lambda_{12}r_{10}(1-r_{12}-r_{21})}{1-\lambda_{12}r_{10}r_{12}} \equiv -\frac{\det \hat{C}_2}{\det \hat{D}_2^0} \tag{6a}$$

$$R_{12}^{(2)} = \frac{G_1^{(1)}\dot{G}_2^{(0)}(x_2+0,x_2) - G_2^{(0)}\dot{G}_1^{(1)}(x_2+0,x_2)}{G_1^{(1)}\dot{G}_2^{(0)}(x_2+0,x_2) - G_2^{(0)}\dot{G}_1^{(1)}(x_2-0,x_2)} = \frac{r_{12}(1-\lambda_{12}-r_{10})}{1-\lambda_{12}r_{10}r_{12}} \tag{6b}$$

with $G_2^{(0)} \equiv G_2^{(0)}(x_2,x_2)$ and $G_1^{(1)} \equiv G_1^{(1)}(x_2,x_2)$. The quantity λ_{12} in equations (6) is defined by

$$\lambda_{12} = \lambda_{21} = \frac{G_1^{(0)}(x_1,x_2)G_1^{(0)}(x_2,x_1)}{G_1^{(0)}(x_1,x_1)G_1^{(0)}(x_2,x_2)} = \exp\left(-\int_{x_1}^{x_2}\frac{dx}{G_1^{(0)}(x,x)}\right). \tag{7}$$

Here we have used the relation connecting $G(x,x')$ with the one-particle GF at coinciding one-dimensional coordinates $x = x'$ [19]:

$$G(x,x') = [G(x,x)G(x',x')]^{1/2}\exp\left(-\int_{\min(x,x')}^{\max(x,x')}\frac{dx_1}{2G(x_1,x_1)}\right)$$

$$= \tfrac{1}{2}[-\dot{\theta}(x')\dot{\theta}(x)]^{-1/2}\exp[i|\theta(x)-\theta(x')|] \tag{8}$$

where

$$G(x,x;E) \equiv i/[2\dot{\theta}(x;E)] \tag{8a}$$

and

$$\theta(x;E) = \int\frac{i\,dx}{2G(x;x;E)} \tag{8b}$$

are phase functions. The coefficients r_{12} and r_{21} are obtained from equation (4) by replacing in the lower indices $0 \to 1$, $1 \to 2$ and $x_1 \to x_2$.

The reflection amplitude $R_{01}^{(2)}$ from the boundary $x = x_1$ may be calculated as follows. Let us build the GF in the region $x, x' \leqslant x_1$ with the second boundary at $x = x_2$. It is not difficult to see that it will have the same form as the GF in equation (1*a*) with the substitution of r_{01} for $R_{01}^{(2)}$:

$$G_0^{(2)}(x,x') = G_0^{(0)}(x,x') - R_{01}^{(2)}\frac{G_0^{(0)}(x,x_1)G_0^{(0)}(x_1,x')}{G_0^{(0)}(x_1,x_1)} \qquad x,x' \leqslant x_1. \tag{9}$$

Evaluating this GF and the GF of equation (5*b*) at $x = x' = x_1$ and solving the linear equation we obtain

$$R_{01}^{(2)} = \frac{r_{01} + \lambda_{12}(1-r_{01}-r_{10})r_{12}}{1-\lambda_{12}r_{10}r_{12}} \equiv -\frac{\det \hat{A}_2}{\det \hat{D}_2^0}. \tag{10}$$

Thus, after having added a new boundary at the point $x = x_2$, we have not only calculated the amplitude $R_{01}^{(2)}$, but also that of the reflected wave $R_{21}^{(2)}$, when the incident wave falls onto the sample from the left.

It is likewise useful to note that denominator and numerator of (10) and (6*a*) may be written in the form:

$$\det \hat{D}_2^0 = \begin{vmatrix} 1 & r_{12}\lambda_{12}^{1/2} \\ r_{10}\lambda_{12}^{1/2} & 1 \end{vmatrix} \tag{10a}$$

$$\det \hat{A}_2 = \left|\begin{array}{c|cc} 0 & r_{01} & r_{12}\lambda_{12}^{1/2} \\ \hline 1 & \multicolumn{2}{c}{\multirow{2}{*}{$\hat{D}_2^0$}} \\ \lambda_{12}^{1/2} & & \end{array}\right| \tag{10b}$$

$$\det \hat{C}_2 = \left|\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\hat{D}_2^0$}} & \lambda_{12}^{1/2} \\ & & 1 \\ \hline r_{10}\lambda_{12}^{1/2} & r_{21} & 0 \end{array}\right|. \tag{10c}$$

Adding once more a new boundary at the point x_3, the corresponding coefficients $R_{23}^{(3)}$ and $R_{32}^{(3)}$ may be calculated. Further, knowing the explicit form of $R_{23}^{(3)}$ and $R_{32}^{(3)}$ (compare the derivation for $R_{01}^{(2)}$), $R_{01}^{(3)}$ may be calculated when three boundaries are available. For clearness, we write the formula in this case:

$$R_{01}^{(3)} = -\det \hat{A}_3 / \det \hat{D}_3^0$$

where

$$D_3^0 \equiv \det \hat{D}_3^0 = \begin{vmatrix} 1 & r_{12}\lambda_{12}^{1/2} & r_{23}\lambda_{13}^{1/2} \\ r_{10}\lambda_{12}^{1/2} & 1 & r_{23}\lambda_{23}^{1/2} \\ r_{10}\lambda_{13}^{1/2} & r_{21}\lambda_{23}^{1/2} & 1 \end{vmatrix} \tag{11}$$

$$\det \hat{A}_3 = \left|\begin{array}{c|cc} 0 & r_{01} & r_{12}\lambda_{12}^{1/2} \quad r_{23}\lambda_{13}^{1/2} \\ \hline 1 & \multicolumn{2}{c}{\multirow{3}{*}{$\hat{D}_3^0$}} \\ \lambda_{12}^{1/2} & & \\ \lambda_{13}^{1/2} & & \end{array}\right|. \tag{11a}$$

Here

$$\lambda_{13} = \lambda_{31} = \lambda_{12}\lambda_{23} = \exp\left(-\int_{x_1}^{x_2} \frac{\mathrm{d}x}{G_1^0(x,x)} - \int_{x_2}^{x_3} \frac{\mathrm{d}x}{G_2^0(x,x)}\right)$$

$$= \exp\left(-\sum_{i=1}^{2} \int_{x_i}^{x_{i+1}} \frac{\mathrm{d}x}{G_i^0(x,x)}\right). \tag{11b}$$

Thus, each nth element of column k of the determinant of D_3^0 is the product of phase coefficient $\lambda_{kn}^{1/2} = \lambda_{nk}^{1/2}$ with the amplitude of reflection coefficient $r_{k,k-1}$ when the wave propagates from the region k into $k-1$ (at $k < n$) and $r_{k-1,k}$ at $k > n$. The diagonal elements of determinant D_3^0 are equal to 1.

Let us now generalize the procedure indicated above in order to derive the amplitudes $R^{(N)}_{N,N-1}$ and $R^{(N)}_{01}$. With the addition of new boundaries at the points $x_4, \ldots, x_{N-1}$, x_N, we shall obtain the GF in the interval $[x_{N-1}, x_N]$, when N boundaries are present, as

$$G^{(N)}_{N-1}(x,x') = G^{(N-1)}_{N-1}(x,x') - R^{(N)}_{N-1,N}\frac{G^{(N-1)}_{N-1}(x,x_N)G^{(N-1)}_{N-1}(x_N,x')}{G^{(N-1)}_{N-1}(x_N,x_N)} \qquad x_{N-1} \leqslant x, x' \leqslant x_N. \quad (12a)$$

Here $R^{(N)}_{N-1,N}$ is the amplitude of reflection from the Nth boundary with all remaining $(N-1)$ boundaries available (compare with equation (6*b*)):

$$R^{(N)}_{N-1,N} = \frac{r_{N-1,N}(1 - R^{(N-1)}_{N-1,N-2}\lambda_{N-1,N})}{1 - \lambda_{N-1,N}r_{N-1,N}R^{(N-1)}_{N-1,N-2}}. \quad (12b)$$

The GF $G^{(N-1)}_{N-1}(x,x')$ in equation (12*a*) has the form:

$$G^{(N-1)}_{N-1}(x,x') = G^{(0)}_{N-1}(x,x') - R^{(N-1)}_{N-1,N-2}\frac{G^{(0)}_{N-1}(x,x_{N-1})G^{(0)}_{N-1}(x_{N-1},x')}{G^{(0)}_{N-1}(x_{N-1},x_{N-1})} \qquad x_{N-1} \leqslant x, x' \leqslant x_N. \quad (12c)$$

On the right-hand side of the Nth boundary the GF has the form:

$$G^{(N)}_N(x,x') = G^{(0)}_N(x,x') - R^{(N)}_{N,N-1}\frac{G^{(0)}_N(x,x_N)G^{(0)}_N(x_N,x')}{G^{(0)}_N(x_N,x_N)} \qquad x, x' \geqslant x_N. \quad (13)$$

Here $R^{(N)}_{N,N-1}$ is the amplitude of electron reflection from the Nth boundary in multilayered structure, when the electron falls in from the right:

$$R^{(N)}_{N,N-1} = \frac{r_{N,N-1} + R^{(N-1)}_{N-1,N-2}\lambda_{N-1,N}(1 - r_{N,N-1}r_{N-1,N})}{1 - \lambda_{N-1,N}r_{N-1,N}R^{(N-1)}_{N-1,N-2}}. \quad (14)$$

If the numerator and the denominator in equation (14) are represented in the form of determinants, as is done for $R^{(2)}_{21}$ (see equation 6(*a*)), and substituted again in equation (14) for $R^{(N-1)}_{N-1,N-2}$, and if the procedure is repeated N times, we shall obtain $R^{(N)}_{N,N-1}$ as a function of r_{nk} and $\lambda^{1/2}_{nk}$, in the following form:

$$R^{(N)}_{N,N-1} = \frac{\det \hat{C}_N}{\det \hat{D}^0_N} = -\frac{\det \hat{\tilde{D}}_{N+1}}{\det \hat{D}^0_N} \equiv -\frac{\tilde{D}_{N+1}}{D^0_N} \quad (15)$$

where matrix $\hat{D}^0_N$ is determined by

$$(\hat{D}^0_N)_{nk} = \delta_{nk} + (1 - \delta_{nk})r_{k,k-1}\lambda^{1/2}_{nk} \qquad n \geqslant k \quad (16a)$$

$$(\hat{D}^0_N)_{nk} = \delta_{nk} + (1 - \delta_{nk})r_{k-1,k}\lambda^{1/2}_{nk} \qquad n \leqslant k \quad (16b)$$

with

$$\lambda_{nk} = \lambda_{kn} = \exp\left(-\sum_{i=\min\{n,k\}}^{-1+\max\{n,k\}} \int_{x_i}^{x_{i+1}} \mathrm{d}x \frac{1}{G_i^{(0)}(x,x)}\right) \qquad n,k \geqslant 0 \tag{16c}$$

and matrix $\hat{\tilde{D}}_{N+1}$ is connected with $\hat{D}_N^0$ and given by

$$1 \leqslant n,k \leqslant N \quad (\hat{\tilde{D}}_{N+1})_{n,k} = (\hat{D}_N^0)_{n,k}$$

$$1 \leqslant n \leqslant N \qquad (\hat{\tilde{D}}_{N+1})_{N+1,n} = r_{n,n-1}\lambda_{nN}^{1/2} \quad (\hat{\tilde{D}}_{N+1})_{n,N+1} = \lambda_{nN}^{1/2} \quad (\hat{\tilde{D}}_{N+1})_{N+1,N+1} = 0. \tag{17}$$

The amplitude of reflection of the electron $R_{01}^{(N)}$ from the first boundary of the layered structure, when the wave falls in from the left, may be evaluated if the above-described procedure that leads to equations (13) and (14) is repeated in reverse order, i.e. new boundaries are added from the left. Hence we find that $R_{01}^{(N)}$ may also be represented in the form of a ratio of determinants depending on r_{nk} and $\lambda_{nk}^{1/2}$ only:

$$R_{01}^{(N)} = -\frac{\det \hat{A}_N}{\det \hat{D}_N^0} \equiv -\frac{\det \hat{\tilde{D}}_{N+1}^0}{\det \hat{D}_N^0} = -\frac{\tilde{D}_{N+1}^0}{D_N^0}. \tag{18}$$

Here matrix $\hat{\tilde{D}}_{N+1}^0$ is connected with $\hat{D}_N^0$ and one gets

$$1 \leqslant n,k \leqslant N \quad (\hat{\tilde{D}}_{N+1}^0)_{n+1,k+1} = (\hat{\tilde{D}}_N^0)_{n,k}$$

$$2 \leqslant n \leqslant N+1 \quad (\hat{\tilde{D}}_{N+1}^0)_{n,1} = \lambda_{n-1,1}^{1/2} \quad (\hat{\tilde{D}}_{N+1}^0)_{1,n} = r_{n-2,n-1}\lambda_{1,n-1}^{1/2} \quad (\hat{\tilde{D}}_{N+1}^0)_{1,1} = 0. \tag{19}$$

For $x, x' \leqslant x_1$ the electron GF has the form:

$$G_0^{(N)}(x,x') = G_0^{(0)}(x,x') - R_{01}^{(N)} \frac{G_0^{(0)}(x,x_1)G_0^{(0)}(x_1,x)}{G_0^{(0)}(x_1,x_1)} \qquad x,x' \leqslant x_1. \tag{20}$$

A recurrence relation for the determinants $\tilde{D}_{N+1}$, $\tilde{D}_{N+1}^0$ and D_N^0, which determine the reflection amplitudes $R_{01}^{(N)}$ and $R_{N,N-1}^{(N)}$ can be derived. Let us state here the recurrence relation for $\tilde{D}_{N+1}$ as an example. It may be obtained using equations (16) and (17). For this purpose it is convenient to decompose the determinant in terms of elements of the last line. As a result, we have

$$D_N^0 = A_N D_{N-1}^0 - B_N D_{N-2}^0$$

$$\tilde{D}_{N+1} = \frac{1 - r_{N,N-1} - r_{N-1,N}}{r_{N-1,N}} D_N^0 - \frac{(1 - r_{N,N-1})(1 - r_{N-1,N})}{r_{N-1,N}} D_{N-1}^0 \tag{20a}$$

where

$$A_1 = 0$$

$$A_N = 1 + \lambda_{N-1,N} \frac{r_{N-1,N}}{r_{N-2,N}}(1 - r_{N-2,N-1} - r_{N-1,N-2}) \qquad N > 1$$

$$B_N = \lambda_{N-1,N} \frac{r_{N-1,N}}{r_{N-2,N-1}}(1 - r_{N-2,N-1})(1 - r_{N-1,N-2})$$

and $D_{N-1(N-2)}^0$ is the determinant (equation (16)) in which the Nth or $(N-1)$th line and column are absent.

Let us finally remark that the quantity $R_{01}^{(N)}$ (also $R_{N,N-1}^{(N)}$) is the amplitude of electron reflection from a multilayered structure, and its is connected with the resistance ρ of the Landauer formula [20].

The procedure described above also permits us to calculate the GF in an arbitrary interval $[x_n, x_{n+1}]$ of a multilayered structure.

Let us break up the layered structure with N boundaries into two blocks, a left block containing $(n-1)$ boundaries and a right block consisting of $(N-n)$ boundaries. The GF in the interval on the left-hand side of the boundary at $x = x_n$, is given by

$$G_{n-1}^{(N)}(x,x') = G_{n-1}^{(n-1)}(x,x') - \tilde{R}_{n-1,n}^{(N)} \times \frac{G_{n-1}^{(n-1)}(x,x_n)G_{n-1}^{(n-1)}(x_n,x')}{G_{n-1}^{(n-1)}(x_n,x_n)} \qquad x_{n-1} \leqslant x, x' \leqslant x_n \tag{21}$$

where $G_{n-1}^{(n-1)}(x,x')$ is yielded by equation (13) with the substitution of $N \to (n-1)$ for lower indices and $0 \to (n-1)$ for upper indices. Here and below, the tilde mark signifies that the given quantity is calculated in the presence of all boundaries from the left and the right.

The quantity $R_{n-1}^{(n-1)}$ in GF $G_{n-1}^{(n-1)}(x,x')$ is the amplitude of electron reflection from the left block (when the electron falls in this block from the right) and is obtained from equation (15) by deleting from D_N^0 the last $N-(n-1)$ lines and $N-(n-1)$ columns:

$$R_{n-1,n-2}^{(n-1)} = -\frac{\det \hat{\tilde{D}}_n}{\det \tilde{D}_{n-1}^0} \equiv -\frac{\tilde{D}_n}{D_{n-1}^0}. \tag{22}$$

Thus, $R_{n-1,n-2}^{(n-1)}$ is the reflection coefficient from the $(n-1)$ boundaries of the left block, when the wave falls from the vacuum on the right.

The electron GF of the right block, containing $(-n+N)$ boundaries, has the form

$$G_n^{(N)}(x,x') = G_n^{(n)}(x,x') - \tilde{R}_{n,n-1}^{(N)} \frac{G_n^{(n)}(x,x_n)G_n^{(n)}(x_n,x')}{G_n^{(n)}(x_n,x_n)} \qquad x_n \leqslant x, x' \leqslant x_{n+1} \tag{23}$$

where $G_n^{(n)}(x,x')$ has a structure analogous to equation (20) with the substitution of $(N) \to (n)$, $0 \to n$, $1 \to n+1$ in the lower indices.

With this, $G_n^{(n)}(x,x')$ depends on $R_{n,n+1}^{(-n+N)}$, which are reflection amplitudes from the nth right block boundary, when the wave falls in from the vacuum on the left-hand side. $R_{n,n+1}^{(-n+N)}$ is obtained from equation (18) by deleting from D_N^0 the first n lines and n columns. We have

$$\begin{aligned} (\hat{D}^0_{-n+N})_{k,m} &= \delta_{km} + (1-\delta_{km}) r_{n+m,m-1+n} \lambda_{n+k,n+m}^{1/2} \qquad k \geqslant m \\ (\hat{D}^0_{-n+N})_{k,m} &= \delta_{km} + (1-\delta_{km}) r_{n+m-1,m+n} \lambda_{n+k,n+m}^{1/2} \qquad k \leqslant m \end{aligned} \tag{24a}$$

so that

$$R_{n,n+1}^{(-n+N)} = -\frac{\det \hat{\tilde{D}}^0_{-n+N+1}}{\det \hat{\tilde{D}}^0_{-n+N}} \equiv -\frac{\hat{\tilde{D}}^0_{-n+N+1}}{\hat{D}^0_{-n+N}}. \tag{24b}$$

Using the condition of continuity for GF (21) and (23) at $x = x_n$ (see equation (2)), we evaluate unknown coefficients $\tilde{R}_{n,n-1}^{(N)}$ and $\tilde{R}_{n-1,n}^{(N)}$ and thus the GF (equations (22)

and (24)) in the regions $[x_{n-1}, x_n]$ and $[x_n, x_{n+1}]$:

$$\tilde{R}^{(N)}_{n,n-1} = \frac{R^{(n)}_{n,n-1}(1 - \lambda_{n,n+1}R^{(-n+N)}_{n,n+1})}{1 - R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1}} \tag{25a}$$

$$\tilde{R}^{(N)}_{n-1,n} = \frac{R^{(-n+1+N)}_{n-1,n}(1 - \lambda_{n-1,n}R^{(n-1)}_{n-1,n-2})}{1 - R^{(-n+1+N)}_{n-1,n}R^{(n-1)}_{n-1,n-2}\lambda_{n-1,n}}. \tag{25b}$$

Concluding this section let us show the expression for the GF at coinciding coordinates:

$$G^{(N)}_n(x,x) = (D^0_N)^{-1}G^{(0)}_n(x,x)(1 + R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1} - R^{(n)}_{n,n-1}\,e^{2i[\theta_n(x)-\theta_n(x_n)]} - R^{(-n+N)}_{n,n+1}\,e^{2i[\theta_n(x_{n+1})-\theta_n(x)]}). \tag{26}$$

Here $\theta_n(x)$ is expressed by equation (8*b*) and we have $D^0_N = 1 - R^{(n)}_{n,n-1}R^{-n+N}_{n,n+1}\lambda_{n,n+1}$.

3. The coefficient of electron transmission through a layered structure

Let us proceed to calculate the coefficient of transparency through a multilayered structure. By means of the definition, it is expressed as the square amplitude of the wavefunction from the right (if the electron falls in from the left) of the given structure and it may be written as

$$T = [|G^{(0)}_0(x_1,x_1)|\,|G^{(0)}_N(x_N,x_N)|]^{-1/2}\,|G^{(N)}(x_1,x_N)|^2 \tag{27}$$

where $G^{(N)}(x_1,x_N)$ is the GF of the electron in the layered structure with N boundaries. Employing equation (8) let us expresss $G^{(N)}(x_1,x_N)$ in the form:

$$G^{(N)}(x_1,x_N) = [G^{(N)}(x_1,x_1)G^{(N)}(x_N,x_N)]^{1/2}\exp\left(-\int_{x_1}^{x_N}\frac{dx}{2G^{(N)}(x,x)}\right)$$
$$= [G^{(N)}_0(x_1,x_1)G^{(N)}_N(x_N,x_N)]^{1/2}\exp\left(-\sum_{n=1}^{N-1}\int_{x_n}^{x_{n+1}}\frac{dx}{2G^{(N)}_n(x,x)}\right). \tag{28}$$

Here $G^{(N)}_N(x,x')$, $G^{(N)}_0(x,x')$ and $G^{(N)}_n(x,x')$ are defined by equations (13), (20) and (26) respectively.

To calculate the integral appearing in equation (28), we shall avail of equation (8*b*), connecting the phase function $\theta(x;E)$ and the GF, as well as equation (26) for $G^{(N)}_n(x,x')$:

$$(-)\int_{x_n}^{x_{n+1}}\frac{dx}{2G^{(N)}_n(x,x)} = \tfrac{1}{2}\ln\frac{\lambda_{n,n+1}(1 - R^{(n)}_{n,n-1})(1 - R^{(-n+N)}_{n,n+1})}{(1 - R^{(n)}_{n,n-1}\lambda_{n,n+1})(1 - R^{(-n+N)}_{n,n+1}\lambda_{n,n+1})}. \tag{29}$$

In view of equations (22), (24) and (29), let us present equation (28) in final form:

$$G^{(N)}(x_1,x_N) = \left(G^{(0)}_0(x_1,x_1)G^{(0)}_N(x_N,x_N)(1 - R^{(N)}_0)(1 - R^{(N)}_{N,N-1}) \times \prod_{n=1}^{N-1}\frac{(1 - R^{(n)}_{n,n-1})(1 - R^{(-n+N)}_{n,n+1})\lambda_{n,n+1}}{(1 - R^{(n)}_{n,n-1}\lambda_{n,n+1})(1 - R^{(-n+N)}_{n,n+1}\lambda_{n,n+1})}\right)^{1/2}$$
$$= (D^0_N)^{-1}\left(G^{(0)}_0(x_1,x_1)G^{(0)}_N(x_N,x_N)\lambda_{1N}\prod_{n=1}^{N}(1 - r_{n,n-1})(1 - r_{n-1,n})\right)^{1/2}. \tag{30}$$

Substituting equation (30) in equation (27) for the GF $G^{(N)}(x_1, x_N)$ and using the analogous expression for $[G^{(N)}(x_1, x_N)]^*$ we shall finally obtain the following compact expression for the coefficient of transparency:

$$T = |D_N|^{-2} \tag{31}$$

where

$$D_N = D_N^0 \left(\lambda_{1N} \prod_{n=1}^{N} (1 - r_{n,n-1})(1 - r_{n-1,n}) \right)^{-1/2}. \tag{31a}$$

Let us recall that the matrix D_N^0 is expressed by equation (16) and $r_{n-1,n}$ and $r_{n,n-1}$ by the analogue of equation (4). In the case of δ-potentials

$$r_{n-1,n} = r_{n,n-1} = \frac{iV_n/2k}{1 + iV_n/2k}$$

when V_n is the amplitude of the nth δ-potential and $E = k^2$. Using these expressions for $r_{n-1,n}$ in equation (31a), we regain the expression for D_N presented earlier in [9, 10].

From the conclusion above, it is clear that expression (31) is easily evaluated for the case when there is a one-dimensional chain of scatterers, characterized by random complex amplitudes of reflection coefficients. Then in the expression for the transmission coefficient (equation (31)) the λ_{ik} are phase multipliers set up by the wave between two scatterers i and k.

4. Propagation of plane sound and electromagnetic waves in layered structures

All general properties of the GF, as well as the formula for the coefficient of transparency T through determinant D_N in the layered structure, obtained in section 3, are valid not only for electrons but also for any waves (sound and electromagnetic), when their propagation through a medium is described by a differential equation of second order.

To evaluate the coefficient of transparency of an electromagnetic wave through the layered structure, it is necessary to express coefficients r_{nk} in equation (31a) by media impedances Z_k [21]:

$$r_{nk} = (Z_k - Z_n)/(Z_k + Z_n) = -r_{kn}. \tag{32}$$

If the vector E of the plane of the electromagnetic wave is perpendicular to the plane of incidence (the plane yz coincides with the boundary of two media, and the plane zx with the plane of the incident wave), the impedance Z_j has the form

$$Z_j \equiv (\mu_j/\varepsilon_j)^{1/2}/\cos\theta_j \tag{33}$$

where θ_j is the incidence angle (or angle of refraction) of the wave at the boundary of two semi-infinite media.

In view of equation (33), we obtain $r_{s(nk)}$ for s-polarized light, when vector E is perpendicular to the plane of incidence, as

$$r_{s(nk)} = \frac{(\varepsilon_n/\mu_n)^{1/2}\cos\theta_n - (\varepsilon_k/\mu_k)^{1/2}\cos\theta_k}{(\varepsilon_n/\mu_n)^{1/2}\cos\theta_n + (\varepsilon_k/\mu_k)^{1/2}\cos\theta_k}. \tag{33a}$$

If the vector $\boldsymbol{E}$ is in the plane of incidence of the wave (p-polarization), the impedances Z_j and $r_{p(nk)}$ may be presented in the form

$$Z_j \equiv (\mu_j/\varepsilon_j)^{1/2} \cos\theta_j \tag{34}$$

$$r_{p(nk)} = \frac{(\varepsilon_k/\mu_k)^{1/2}\cos\theta_n - (\varepsilon_n/\mu_n)^{1/2}\cos\theta_k}{(\varepsilon_k/\mu_k)^{1/2}\cos\theta_n + (\varepsilon_n/\mu_n)^{1/2}\cos\theta_k}. \tag{34a}$$

The quantity $\lambda_{k,k+1}$ in equation (7), entering equation (31) for D_N, acquires the following form:

$$\lambda_{k,k+1} = \exp\left(-\int_{x_k}^{x_{k+1}} \frac{dx}{G_k^{(0)}(x,x)}\right) = \exp(2i\kappa_k|x_k - x_{k+1}|). \tag{35}$$

In deriving this expression, we have made use of the fact that the GF of the wave equation in a homogeneous medium with dielectric permeability $\varepsilon_k(\omega)$ satisfies the equation

$$(-\partial^2/\partial x^2 + q^2 - \varepsilon_k(\omega)\omega^2/c^2)G_k^{(0)}(x,x';\kappa_k) = \delta(x-x') \tag{36}$$

and has the form:

$$G_k^{(0)}(x,x';\kappa_k) = [i/(2\kappa_k)]\,e^{-i\kappa_k|x-x'|}.$$

Here

$$\kappa_k^2 = \varepsilon_k(\omega)\omega^2/c^2 - q^2$$

and q is the two-dimensional wavevector in the plane zy.

Thus, the determinant D_N (equation (31a)), determining the coefficient of transparency T for the electromagnetic (or acoustic) wave, may be written in the form:

$$D_N = D_N^0\left(\lambda_{1N}\prod_{n=1}^{N}(1 - r_{n,n-1}^2)\right)^{-1/2} \tag{37}$$

since $r_{n-1,n} = -r_{n,n-1}$ (see equation (32)). If $r_{n,n-1}$ are real, then we have

$$D_N = D_N^0\left(\lambda_{1N}\prod_{n=1}^{N}(1 - t_{n-1,n})\right)^{-1/2} \tag{37a}$$

where $t_{n-1,n}$ is the amplitude of the coefficient of transmission through n boundaries.

To close this section let us consider the layered structure representing a random alternation of thin metal layers with a dielectric. Let the metal layers be characterized by a complex dielectric permeability ε_n. The coefficient of light reflection from one metal layer with thickness a_n at normal incidence from the vacuum is (see equation (10))

$$R_{n-1,n}^{(2)} = r_{n-1,n}(1 - e^{2i\kappa_n a_n})/(1 - r_{n-1,n}^2\,e^{2i\kappa_n a_n})$$

where

$$\kappa_0 = \omega/c \qquad \kappa_n = \varepsilon_n^{1/2}\kappa_0 \qquad r_{n-1,n} = (\varepsilon_n^{1/2} - 1)/(\varepsilon_n^{1/2} + 1).$$

If $a_n \to 0$, $\varepsilon_n \to \infty$ and the imaginary part is much smaller than the real part, then we get within the limit $\kappa_0 \varepsilon_n a_n = \text{const} = V + i\gamma$ ($\gamma \ll V$) (i.e. for identical homogeneous layers)

$$R^{(2)}_{n-1,n} = \frac{i(V + i\gamma)/2\kappa_0}{1 + i(V + i\gamma)/2\kappa_0} \tag{38}$$

and the expression for D_N has the same form as in [9, 10], i.e. it coincides with the expression for D_N in the Lloyd model [9, 10] after averaging over the random potentials. Thus, a periodic layered structure of thin metal films is the experimental realization of the Lloyd model in experiments on light transmission.

Sipe *et al* [22] have shown that the radius of light polarization in a random layered structure depends upon the polarization of light and the incidence angle of the wave using Monte Carlo simulation. The authors have framed a simple theory within the long-wave limit; though it displays the main properties qualitatively, it has however no quantitative agreement with the Monte Carlo simulation.

Let us calculate the radius of localization in the model of Sipe *et al* [22], when there are two alternating types of layers with dielectric permeabilities ε_1 and ε_2 ($\mu_{1,2} = 1$) with random thicknesses distributed by the law $P(a) = a_0^{-1} \exp(a/a_0)$. The Fresnel coefficients for s- and p-polarized waves according to (33*a*) and (34*a*) have the form ($n^2 = \varepsilon_2/\varepsilon_1$):

$$r_s = \frac{\cos\theta - (n^2 - \sin^2\theta)^{1/2}}{\cos\theta + (n^2 - \sin^2\theta)^{1/2}} \tag{39}$$

$$r_p = \frac{n^2\cos\theta - (n^2 - \sin^2\theta)^{1/2}}{n^2\cos\theta + (n^2 - \sin^2\theta)^{1/2}}. \tag{39a}$$

If r_s, $r_{s,p} \ll 1$, the determinant D_N may be calculated for an arbitrary distribution in the thicknesses of the layers $r^2_{n-1,n} = r^2_{s,p}$ [23]:

$$D_N^{-1} = e^{i\varphi_{1,N}} \prod_{n=1}^{N} (1 - r^2_{s,p})^{1/2} \simeq e^{i\varphi_{1,N} - Nr^2_{s,p}/2} \tag{40}$$

and the coefficient of transparency is given by

$$T_{s,p} = \exp(-Nr^2_{s,p}). \tag{41}$$

Thus, in this limiting case, the localization length is

$$a_0/l_{s,p} = r^2_{s,p}/2. \tag{42}$$

It follows from expressions (39), (39*a*) and (42) that the localization length becomes infinite at the Brewster angle $\tan\theta_B = n$. The comparison of equation (42) with the results of Monte Carlo simulation (figure 1) shows that equation (42) describes the experimental results well, when $\theta \leqslant 65°$. In the region of large angles, where $(1 - r_{s,p}) \ll 1$, the determinant D_N may also be calculated [23]. We shall not repeat here the calculations presented in [23], but state the final expression for localization length:

$$\frac{a_0}{l_{s,p}} = \tfrac{1}{2}\left\{\ln\frac{(n^2 - \sin^2\theta)^{1/2}}{4\gamma^2 n^2\cos\theta} - \operatorname{Re}\left[\Psi\left(1 + \frac{i}{2\kappa_1 a_0}\right) + \Psi\left(1 + \frac{i}{2\kappa_2 a_0}\right)\right]\right\} \tag{43}$$

where $\Psi(x)$ is the di-gamma function, $\kappa_{1,2} = \kappa_0 \varepsilon_{1,2}^{1/2}$ and $\ln\gamma = C = 0.577\ldots$.

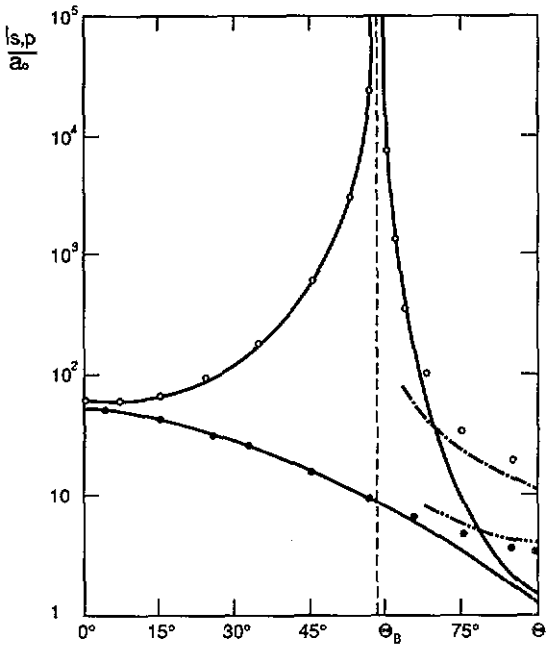

Figure 1. The dependence of the localization length on the angle of light incidence at $\lambda = 5a_0$, $n^2 = \varepsilon_2/\varepsilon_1 = 2.42$. Open circles are the results of Monte Carlo simulations for p-polarized light, and the full circles are for s-polarized light according to [22]. Full curves are calculated from equation (42). The chain curves display the results of equation (43).

The comparison of equation (43) in the region of large angles of incidence with the Monte Carlo simulation [22] is shown in figure 1. It is seen that equation (43) describes the Monte Carlo results well for $\theta > 70°$.

5. Surface waves in a layered structure

Besides the considered questions connected with the calculation of the coefficient of wave transparency through a one-dimensional random system, knowledge of the explicit form of determinant D_N allows us to study the energy spectrum of excitations in the layered structure and also their propagation (surface polaritons and plasmons, etc). For this, it is necessary to study the zeroes of determinant D_N in equation (31*a*), or, what is the same, of the pole $G_n^{(N)}(x, x)$ in equation (26).

Let us consider, for instance, a sandwich: a one-layered film, placed between two semi-infinite media. From equation (31*a*), we get ($N = 2$, the number of boundaries)

$$D_2 = \begin{vmatrix} 1 & r_{12}\, e^{i\kappa_1 a_1} \\ r_{10}\, e^{i\kappa_1 a_1} & 1 \end{vmatrix} = 1 - r_{10} r_{12}\, e^{2i\kappa_1 a_1} = 0. \tag{44}$$

Here (see equations (4*a*) and (34*a*)) one has

$$r_{10} = \frac{G_1^{(0)} - G_0^{(0)}}{G_1^{(0)} + G_0^{(0)}} = \frac{Z_0 - Z_1}{Z_0 + Z_1} = \frac{i\varepsilon_0\kappa_1 - \varepsilon_1\kappa_0}{i\varepsilon_0\kappa_1 + \varepsilon_1\kappa_0}$$

$$r_{12} = \frac{G_1^{(0)} - G_2^{(0)}}{G_1^{(0)} + G_2^{(0)}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{i\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2}{i\varepsilon_2\kappa_1 + \varepsilon_1\kappa_2}.$$

Let us note that GF $G_0^{(0)}$ and $G_2^{(0)}$ are chosen to be real, which signifies attenuation on both sides of the film (there is no incident wave on the left, and no propagating wave on the right).

Equation (44) has a solution at real ε_n ($n = 0, 1, 2$) only for imaginary values κ_1 and presents the known dispersion relation for surface polaritons (see, for example, Maradudin [24]).

For example, if we choose ε_1 for the film in the form $\varepsilon_1 = 1 - \omega_p^2/\omega^2$ (ω_p is the plasma frequency) and $\varepsilon_0 = \varepsilon_2 = 1$ (vacuum) and solve equation (44), we obtain the well known symmetric and antisymmetric modes of surface plasmons in metal films [24].

6. Electron transmission through a magnetic structure

Let us consider the transmission of an electron through the structure, containing ferromagnetic layers, in which the magnetization vectors are always collinear. The coefficient of transparency, and also that of reflection, through such a layer depend on the relative orientation of magnetization vectors in the layer and the spin direction. If the electron spin does not flip on passing through the ferromagnetic layer, this problem is equivalent to the propagation of two independent modes of the electron wave with opposite spins through the sample, and the full coefficient of transparency is given by

$$T = \tfrac{1}{2}(T_\uparrow + T_\downarrow). \tag{45}$$

The coefficient of α-spin electron reflection from each barrier comprises two parts: a part depending on the magnetization vector in the ferromagnetic layer $\delta r_\alpha(M_i)$ and a part not depending on the spin r:

$$r_\alpha(M_i) = r + \delta r_\alpha(M_i).$$

In the case of strong scattering, when $1 - r_\alpha(\{M_i\}) \simeq \Delta + \delta r_\alpha \ll 1$ the determinant $D_N(\{M_i\})$ is easily calculated as

$$D_N^{-2}(\{M_i\}) = [D_N^{(0)}(0)]^{-2}\lambda_{1N} \exp\left(-\sum_{n=1}^{N} \ln[\Delta + \delta r_\alpha(M_i)] - N\ln 2\right). \tag{46}$$

If $\delta r_\alpha(M_i) \ll \Delta$, then with the inclusion

$$\langle \delta r_\alpha \rangle = \frac{1}{N}\sum_{n=1}^{N} \delta r_\alpha(M_i)$$

we obtain

$$D_N^{-2}(\{M_i\}) = D_N^{-2}(0) \exp(-N\langle \delta r_\alpha \rangle/\Delta). \tag{47}$$

Using equation (47) and the connection of the conductance with the permeability of

a barrier, $G(\{M_i\}) = \Sigma_\alpha T_\alpha$ when measured by the two-contact method [25], we find that the conductance part, depending on magnetization, is

$$\Delta G(M_i)/G(0) = \tfrac{1}{2}\sum_\alpha \mathrm{e}^{-N\langle \delta r_\alpha\rangle/\Delta} - 1.$$

Spin polarization appears when the electrons pass through the barrier since the permeability is different for various spin directions. One has

$$\rho_2^{(N)} = (T_\uparrow - T_\downarrow)/(T_\uparrow + T_\downarrow) = \tanh N(\langle \delta r_\uparrow\rangle - \langle \delta r_\downarrow\rangle)/2\Delta. \tag{48}$$

It is seen from equation (48) that the full polarization of the electrons appears at large distances, independently of whether the sample has full magnetization or not.

7. Density of states in layered structures

Knowing the explicit form of the electron GF (26) in the layered structure, it is not difficult to calculate the local density of states at the point x. This expression is the generalization of the formula obtained earlier by one of the authors for random δ-potentials [10].

Using recurrence relations (section 2) and carrying out similar calculations [10], we get

$$\begin{aligned}\nu_n(E;x) = \frac{\mathrm{Im}}{\pi} G_n^{(N)}(x,x;E) = \frac{1}{\pi}\,\mathrm{Im}\,\frac{G_n^{(0)}(x,x)}{D_N^0[G_n^{(0)}(x_n,x_n)]^2}\\ \times\Big\{(D_n^0 - D_{n-1}^0)(D_{-(n-1)+N}^0 - D_{-n+N}^0)\\ - G_n^{(0)}(x_n,x_n)\Big[\Big(1-\cos\int_{x_n}^{x}\frac{\mathrm{i}\,\mathrm{d}t}{G_n^{(0)}(t,t)}\Big)[2G_n^{(0)}(x_n,x_n)D_n^0 D_{-n+N}^0\\ - (D_n^0 - D_{n-1}^0)D_{-n+N}^0 - (D_{-(n-1)+N}^0 - D_{-n+N}^0)D_n^0]\\ -\mathrm{i}\sin\int_{x_n}^{x}\frac{\mathrm{i}\,\mathrm{d}t}{G_n^{(0)}(t,t)}[(D_n^0 - D_{n-1}^0)D_{-n+N}^0\\ - (D_{-(n-1)+N}^0 - D_{-n+N}^0)D_n^0]\Big]\Big\}.\end{aligned} \tag{49}$$

The density of states averaged by the thickness of the given layer is

$$\nu_n(E) = \frac{1}{\pi a_n}\int_{x_n}^{x_{n+1}} \mathrm{Im}\, G_n^{(N)}(x,x;E)\,\mathrm{d}x \tag{50}$$

and may also be expressed by determinant D_N. In order to derive this, let us insert the explicit expression for the GF $G_n^{(N)}(x,x;E)$ (expression (26)) into (50) and write the resulting expression as a sum of three terms, each of which may be exactly calculated, not exploiting the explicit form [19]:

$$\begin{aligned}I_1 &= \frac{1 + R_{n,n-1}^{(n)} R_{n,n+1}^{(-n+N)}\lambda_{n,n+1}}{1 - R_{n,n-1}^{(n)} R_{n,n+1}^{(-n+N)}\lambda_{n,n+1}}\int_{x_n}^{x_{n+1}} G_n^{(0)}(x,x)\,\mathrm{d}x\\ &= \frac{1 + R_{n,n-1}^{(n)} R_{n,n+1}^{(-n+N)}\lambda_{n,n+1}}{1 - R_{n,n-1}^{(n)} R_{n,n+1}^{(-n+N)}\lambda_{n,n+1}}\left[\frac{\mathrm{d}}{\mathrm{d}E}\mathrm{i}\theta_n(x,E) - \frac{G_n^{(0)}}{2}\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{\dot G_n^{(0)}}{G_n^{(0)}}\right)\right]\Bigg|_{x_n}^{x_{n+1}}\end{aligned}$$

$$I_2 = -\frac{R^{(n)}_{n,n-1}\,\mathrm{e}^{-2i\theta_n(x_n;E)}}{1-R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1}}\int_{x_n}^{x_{n+1}} G^{(0)}_n(x,x)\,\mathrm{e}^{+2i\theta_n(x)}\,\mathrm{d}x$$

$$= -\frac{R^{(n)}_{n,n-1}\,\mathrm{e}^{-2i\theta_n(x_n;E)}}{1-R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1}}\left[-\frac{G^{(0)}_n}{2}\mathrm{e}^{+2i\theta_n(x_n;E)}\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{\dot G^{(0)}_n-1}{G^{(0)}_n}\right)\right]\Bigg|_{x_n}^{x_{n+1}}$$

$$I_3 = -\frac{R^{(-n+N)}_{n,n+1}\,\mathrm{e}^{2i\theta_n(x_{n+1};E)}}{1-R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1}}\int_{x_n}^{x_{n+1}} G^{(0)}_n(x,x)\,\mathrm{e}^{-2i\theta_n(x;E)}\,\mathrm{d}x$$

$$= -\frac{R^{(-n+N)}_{n,n+1}\,\mathrm{e}^{2i\theta_n(x_{n+1};E)}}{1-R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1}}\left[-\frac{G^{(0)}_n}{2}(x,x)\mathrm{e}^{-2i\theta_n(x_n;E)}\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{\dot G^{(0)}_n+1}{G^{(0)}_n}\right)\right]\Bigg|_{x_n}^{x_{n+1}}$$

where $\theta_n(x;E)$ is the phase function; it is determined by formula (8*b*). After simple transformations, the following expression for $\nu_n(E)$ is obtained:

$$\nu_n(E) = \frac{1}{\pi a_n}\mathrm{Im}(I_1+I_2+I_3) = \frac{1}{\pi a_n}\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}E}$$

$$\times \ln\frac{1-R^{(n)}_{n,n-1}R^{(-n+N)}_{n,n+1}\lambda_{n,n+1}}{[\lambda_{n,n+1}(1-R^{(n)}_{n,n-1})(1-R^{(n)}_{n-1,n})(1-R^{(-n+N)}_{n,n+1})(1-R^{(-n+N)}_{n+1,n})]^{1/2}}. \tag{51}$$

Substituting equations (22) and (24) into (51), the final expression for $\nu_n(E)$ is gained in the form:

$$\nu_n(E) = \frac{1}{\pi a_n}\mathrm{Im}\left(\frac{\partial}{\partial E}\ln D_N\right)_n. \tag{52}$$

Here the subscript n means that the derivative with respect to the energy has to be calculated upon the quantities characterizing the nth subsystem—i.e. phase function $\theta_n(x;E)$ and amplitude of reflection $r_{n,n-1}$ ($r_{n-1,n}$) and $r_{n,n+1}$ ($r_{n+1,n}$) which are contained in $G^{(0)}_n(x,x;E)$.

Equation (52) is important since for the calculation of the density of states in various parts of the layered system there is no necessity to calculate the GF; it is sufficient to know the energy spectrum.

The density of states averaged by the thickness of the layered system is

$$\nu_n(E) = \frac{1}{a}\sum_{n=1}^{N-1} a_n\nu_n(E) = \frac{1}{\pi a_n}\sum_{n=1}^{N-1}\mathrm{Im}\left(\frac{\partial}{\partial E}\ln D_N\right)_n \tag{53}$$

where $a = x_N - x_1$ is the length of the systems. At $N\to\infty$ the density of states (53) may be represented as [9, 10, 26]

$$\nu_n(E) = \frac{1}{\pi a}\mathrm{Im}\frac{\partial}{\partial E}\ln D_N.$$

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