CALIFORNIA STATE UNIVERSITY, BAKERSFIELD

Lee Webb Math Field Day 2024

Team Medley, Junior-Senior Level

Each correct answer is worth ten points. Answers require justification. Partial credit may be given. Unanswered questions are given zero points.

You have 50 minutes to complete the Exam. When the exam is over, give only one set of answers per team to the proctor. Multiple solutions to the same problem will invalidate each other.

Elegance of solutions may affect score and may be used to break ties.

All calculators, cell phones, music players, and other electronic devices should be put away in backpacks, purses, pockets, etc. Leaving early or otherwise disrupting other contestants may be cause for disqualification.

1. A group of 11 scientists are working on a secret project, for which the materials are kept in a special box that has room for several locks. They want to be able to open the box only when a majority of the scientists are present. How many locks will they need and how many keys does each scientist need?

   Solution: For any group of 5 scientists there needs to be a lock that that group does not have a key for. Since there are $11C5=\binom{11}{5}=\frac{11\cdot10\cdot9\cdot8\cdot7}{5\cdot4\cdot3\cdot2}=462$ distinct groups, this is number of locks required. Suppose Scientist A is joined by 5 others. Together they form a majority – they all have to have keys to all the locks. Since there are $10C5=\binom{10}{5}=\frac{10\cdot9\cdot8\cdot7\cdot6}{5\cdot4\cdot3\cdot2}=252$ such sets of scientists that Scientist A could join, they must all have 252 keys.

2. Find all right triangles that have integer length sides and that have the property that the perimeter equals the area.

   Solution: Suppose we have such a triangle and the lengths of the legs are x and y. Then the area is xy/2 and the perimeter is $x+y+\sqrt{(x^2+y^2)}$. Setting these equal to each other, we obtain:

$$x+y+\sqrt{(x^2+y^2)}=\frac{1}{2}xy$$

$$\sqrt{(x^2+y^2)}=\frac{1}{2}xy-x-y$$

$$x^2+y^2=(\frac{1}{2}xy-x-y)^2=\frac{1}{4}x^2y^2+x^2+y^2-x^2y-xy^2+2xy$$

$$0=\frac{1}{4}x^2y^2-x^2y-xy^2+2xy$$

$$0=x^2y^2-4x^2y-4xy^2+8xy=xy(xy-4x-4y+8)=xy(x-4)(y-4)-8$$

We can assume x and y are positive, so this reduces to

$$(x-4)(y-4)=8$$.

So, (x-4, y-4) equal (8,1), (4,2), (2,4), or (1,8). By symmetry, we only need the first two of these. They correspond to (x,y) equal (12, 5) and (8,6). Put another way, the only such triangles are the one that have sides 5,12, 13 or 6,8,10.

3. The function $f(x)=x2^x$ is increasing for positive values of *x*. Therefore this function has an inverse. Let $G(x)$ be this inverse function. In terms of *G(x)*, solve the equation $2^x+x=5$.

Solution:

$$2^x+x=5$$

$$1+x2^{-x}=5\cdot 2^{-x}$$

$$1=(5-x)2^{-x}$$

$$32=(5-x)2^{5-x}$$

$$G(32)=5-x$$

$$x=5-G(32)$$

4. For a triangle ABC, we have AB =1, AC = 2 and the difference between the measurements of angles B and C is $\frac{2\pi}{3}$. What is the area of the triangle?

Solution: We will write the angles in terms of C. E.g. $B=C+2\pi/3$ and $A=\pi-B-C=\pi-(C+2\pi/3)-C=\pi/3-2C$.

The area of a triangle is ½ the product of the lengths of two sides and the sine of the enclosed angle. In this case, we have Area=

$$\sin(A)=\sin(\pi/3-2C)=\frac{\sqrt{3}}{2}\cos(2C)-\frac{1}{2}\sin(2C)=\frac{\sqrt{3}}{2}(\cos^2 C-\sin^2 C)-\frac{1}{2}2\sin C\cos C.$$

The Law of Sines, written as $\frac{\sin B}{AC}=\frac{\sin C}{AB}$ gives $\frac{\sin B}{2}=\frac{\sin C}{1}$. Cross-multiplying gives $2\sin C=\sin(C+2\pi/3)$. With the sine of a sum formula (also used above) this give $2\sin C=\sin C(\frac{-1}{2})+\cos C(\frac{\sqrt{3}}{2})$. So, $\frac{5}{2}\sin C=\frac{\sqrt{3}}{2}\cos C$ and then $\tan C=\frac{\sqrt{3}}{5}$. Since C is, obviously, an acute angle, from this we can determine sinC and cosC without ambiguity. $\sin C=\sqrt{\frac{3}{28}}$ $\cos C=\frac{5}{\sqrt{28}}$.

Substituting these values into the formula for the area above gives:

Area = $\frac{\sqrt{3}}{2}(\frac{25}{28}-\frac{3}{28})-\sqrt{\frac{3}{28}}\frac{5}{\sqrt{28}}=\frac{3\sqrt{3}}{14}$