CALIFORNIA STATE UNIVERSITY, BAKERSFIELD

Lee Webb Math Field Day 2024

Team Medley, Freshman-Sophomore Level

Each correct answer is worth ten points. Answers require justification. Partial credit may be given. Unanswered questions are given zero points.

You have 50 minutes to complete the Exam. When the exam is over, give only one set of answers per team to the proctor. Multiple solutions to the same problem will invalidate each other.

Elegance of solutions may affect score and may be used to break ties.

All calculators, cell phones, music players, and other electronic devices should be put away in backpacks, purses, pockets, etc. Leaving early or otherwise disrupting other contestants may be cause for disqualification.

1. Six colors of paint are available. Each face of a cube is to be painted a different color. In how many ways can this be done, if two colorings are considered the same if one can be obtained by rotating the other?

   Solution: Say the colors are Red, Blue, Green, Yellow, Orange, and Purple.

   Imagine that a cube has been painted. No matter how it is painted, it can be held so that Red is in the front. Then there are 5 possibilities for which color is in the back. Say the back color is Blue. Then the cube can be rotated so that Green is on top. There are then 3 faces unspecified, so there are 3! = 6 ways of coloring these. Thus, there are 5x6 = 30 ways to paint the cube.

2. How many integers solutions are there to the equation: $|x|+|y|\leq 100$?

   Solution: If x=0, there are 201 choices for y. Likewise, if y=0, there are 201 choices for x. Since x=y=0 is counted in both these lists, we can say that there are 401 solutions where at least one of the variables equals 0. For solutions in the first quadrant, we must have 1<=x<=99. For each such x, there are 100-x possibilities for y. So the number of first quadrant solutions is 99+98+97+...+1= 100x99/2 = 50x99 = 4950. The number of solutions in each quadrant is the same, so the total is 4950x4 + 401 = 20201.

3. Find all right triangles that have integer length sides and that have the property that the perimeter equals the area.

   Solution: Suppose we have such a triangle and the lengths of the legs are x and y. Then the area is xy/2 and the perimeter is $x+y+\sqrt{(x^2+y^2)}$. Setting these equal to each other, we obtain:

$$x+y+\sqrt{(x^2+y^2)}=\frac{1}{2}xy$$

$$\sqrt{(x^2+y^2)}=\frac{1}{2}xy-x-y$$

$$x^2+y^2=(\frac{1}{2}xy-x-y)^2=\frac{1}{4}x^2y^2+x^2+y^2-x^2y-xy^2+2xy$$

$$0=\frac{1}{4}x^2y^2-x^2y-xy^2+2xy$$

$$0=x^2y^2-4x^2y-4xy^2+8xy=xy(xy-4x-4y+8)=xy(x-4)(y-4)-8$$

We can assume x and y are positive, so this reduces to

$(x-4)(y-4)=8$.

So, (x-4, y-4) equal (8,1), (4,2), (2,4), or (1,8). By symmetry, we only need the first two of these. They correspond to (x,y) equal (12, 5) and (8,6). Put another way, the only such triangles are the one that have sides 5,12, 13 or 6,8,10.

4. Determine which is greater: $\sqrt[99]{99!}$ or $\sqrt[100]{100!}$

Solution: Let $x=\sqrt[99]{99!}$ and $y=\sqrt[100]{100!}$.

Then note $\left(\frac{x}{y}\right)^{9900}=\frac{(99!)^{100}}{(100!)^{99}}=\frac{(99!)^{99}99!}{(99!)^{99}100^{99}}=\frac{1}{100}\frac{2}{100}\frac{3}{100}\cdots\frac{99}{100}<1$. So, x<y.