CALIFORNIA STATE UNIVERSITY, BAKERSFIELD

Lee Webb Math Field Day 2013

Team Medley, Freshman-Sophomore Level

Each correct answer is worth ten points. Answers require justification. Partial credit may be given. Unanswered questions are given zero points.

You have 50 minutes to complete the Exam. When the exam is over, give only one set of answers per team to the proctor. Multiple solutions to the same problem will invalidate each other.

Elegance of solutions may affect score and may be used to break ties.

All calculators, cell phones, music players, and other electronic devices should be put away in backpacks, purses, pockets, etc. Leaving early or otherwise disrupting other contestants may be cause for disqualification.

1. Fill in the following array, so that each row and column form an arithmetic sequence (e.g. a sequence in which successive terms all have the same difference):

4d				
3d	74			
2d	y			186
d	x	103		
0				

Solution: First, fill in the spaces as shown above. Then we have $x=\frac{d+103}{2}$ and $y=\frac{x+74}{2}$. Combining these gives $y=\frac{(d+103)/2+74}{2}=\frac{d+251}{4}$. Let $D=y-2\text{d}=\frac{251-7\text{d}}{4}$. The third row gives 2d+4D=186. Thus, $2\text{d}+251-7\text{d}=251-5\text{d}=186$. This gives d=13. We can fill in the table as follows:

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

2. Find the last three digits of the product of the first 20 prime numbers.

Solution: The answer is 390. I checked this the long way with SAGE, but I also got this answer first , without too much work, by hand. First note that primes 2 and 5 will give a zero at the end. Therefore with the other 18 primes, only the last two digits have to be kept track of. Pairing them wisely saves some effort. E.g. 29*31=899 is congruent to -1 (mod 100). Likewise for 59*61.

3. Two parallel lines are 10 units apart. Point A is between the lines and 3 units from the nearer one. Find the area of the largest square that has point A as one vertex and does not contain any points outside the two lines.

Solution: Let the lines be the x-axis and the line y=10. A is the point (0,3,). It is clear that to be optimal the two vertices of the square that are adjacent to A should be on the lines. Let these points be B and C, located at (b,0) and (c,10). Since AB and AC are perpendicular, their slopes are negative reciprocals. This gives bc=21. Also, AB and AC are equal in length, so $9+b^2=49+c^2$. (Note this quantity is also equal to the area of the square. This simplifies to $\left(\frac{21}{c}\right)^2=40+c^2$ or $b^4+40\mathrm{b}^2-21^2$. The quadratic formula gives $c^2=\frac{-40+\sqrt{40^2+4*21^2}}{2}=-20+\sqrt{400+441}$. So the area of the square is $29+\sqrt{841}$.

4. Ella and Daniel are preparing to play a game called "Twenty-One Stones". The 21 stones will be placed in 2 piles. Daniel and Ella will alternate turns. On his or her turn, each player is to remove as many stones as he or she wants from either pile or remove the same number from both piles. The player to take the last stone wins. Ella says since she is younger, she gets to go first. Daniel says "fine – but I get to divide the stones into piles." How many stones should Daniel put into each pile so that he can guarantee that he wins the game?

Solution: Consider the set of safe positions for Daniel to leave Ella. Obviously (0,0) is safe, since it means Daniel has already won. What is the simplest position from which Ella could not win – clearly (1,2) – this is the next safe position. What is the next position up from which Ella could not leave Daniel with (1,2). This is (3,5). After this comes (4,7), (6,10), and (8,13). The pattern would continue with each pair having a difference one greater than the previous pair and each first number being one or two greater than the previous first number – depending on whether or not the number one greater was already part of a safe pair or not. So, we could continue, but we already see the answer is Daniel should make one pile of 8 stones and another pile of 13. To double check, if he picked (10,11) or (9,12), Ella could in one move go all the way to (1,2) or (4,7), respectively. Everyone of Daniel's other choices involves leaving one pile of 7 or lower and Ella likewise could, in a single move, go to one of the safe positions and force a win for herself.

5. A sequence of digits begins with 2,0,1,3 and continues with each subsequent digit being the last digit of the sum of the previous 4 digits. Thus the next few digits in the seqeunce are 6,0,0,9,5,4,8.... Will the string of digits 9988 appear in this sequence?

Solution: With SAGE, I added 500 digits and did not find the string 9988. However, the given rule also allows one to add digits that must have come before. So, if we continued the string 2013 to the left, we find that the sequence could have started 998849902013. Since there are only a finite number of possible 4 digit windows, this string of digits is periodic. Thus, since 9988 came before 2013, it must also be the case that eventually 9988 will come again after 2013.

6. Prove that a convex quadrilateral can be inscribed in a circle if and only if it has opposite angles are that are supplementary.

Solution: First assume that quadrilateral ABCD is inscribed in a circle. Let O be the center of the circle. Then the inscribed angle theorem says that angle ABC has half the measure of angle AOC. Similarly, angle ADC has half the measure of the (other) angle AOC. But the together the two angle AOC's equal 360 degrees, so angles ABC and ADC sum to 180 degrees – i.e. they are supplementary. We could do the same for the other pair of angles of ABCD – or we could just appeal to the fact that the sum of all four angles of a quadrilateral is 360 degrees.

For the converse, we will need the following **fact** (I would call it a lemma, except that I will not prove it here): If P is interior to triangle ABC, then angle ABC is smaller than angle APC. Now, back to the question: let ABCD be a quadrilateral with opposite angles supplementary. There is a circle that goes through points A, B, and C. If D outside the circle, then let E be the intersection of the BD and the circle. Now, E is interior to triangle ACD, so the **fact** says that angle AEC is bigger than angle ADC. But, by the first part of this solution, angle AEC is supplementary to ABC and by hypothesis, so is angle ADC. This is a contradiction. Therefore, D is not outside the circle. If D is inside the circle, we can extend ray BD until it intersects the circle, say at F. Now D is interior to triangle AFC, so the **fact** says that angle ADC is larger than angle AFC. By the first part of this problem angle AFC is supplementary to angle ABC. But, by hypothesis, so is angle ADC. Thus D is not inside the circle. Therefore, D is on the circle that goes through A, B, and C – as was to be proved.